Fritz Laboratory Reports

Civil and Environmental Engineering

1973

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APPLICATIONS OF THE FINITE ELEMENT METHOD TO INELASTIC BEAM-COLUMN PROBLEMS

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March, 1973

Fritz Engineering Laboratory Report No. 400.11

TABLE OF CONTENTS

ABSTRACT

This report contains a pilot study on the application of an incremental iterative Finite Element analysis technique to the inelastic beam-column problem. A layered beam type element is used together with the Ramberg-Osgood law to develop a tangent stiffness matrix formulation. Numerical comparisons are made with the interaction curves previously obtained for two steel and two reinforced concrete beam-columns. Conclusions are drawn as to the place this approach has in the body of knowledge pertaining to inelastic beam-columns.

1. INTRODUCTION

This report contains the results of a pilot study on the application of the Finite Element Method to the inelastic analysis of beam-columns. The objective of the study was to extend an existing formulation and relevant computer program to perform the analysis of beam-columns (Refs. 8 - 11). The existing program performed on inelastic load deflection analysis on non-homogeneous beams using an incremental, iterative, tangent stiffness Finite Element analysis. A layered beam element was used. If the pilot study was successful it would then have been shown that the methodology inherrent in that program could also be applied to beam-columns.

Consistent with the nature of a pilot study only those changes actually necessary to generate preliminary results were effected. The changes in coding were minimal. It was noted during this work that other, more extensive changes to coding and to the iteration techniques employed would enable more efficient and slightly more accurate results to be obtained. Problems associated with iteration schemes and suggested improvements to the existing scheme will be discussed where applicable in Chapters 3, 4 and 5.

Results of numerical investigations on steel and reinforced concrete beam-columns subjected to concentrated midspan lateral loads were compared with existing analysis techniques via interaction curves. The results have shown that the method does provide a new solution technique for inelastic beam-column problems. This method is based on the assumption that plane sections is an adequate representation of the cross-sectional strain field and that the cross section is symmetric about the plane of the lateral load. Failure is assumed to occur in the plane of bending. Within these assumptions the method is not inherrently restricted to cross section, material stress-strain curve, type of loading, or boundary conditions.

The load deflection curve of a given beam-column is also obtained as part of the analysis. This load deflection curve approaches, but does not extend past, the peak of the curve.

2. PREVIOUS WORK

2.1 Introduction

The study of inelastic beam-columns has proceeded along such diverse paths, as electrical analogy, numerical integration and attempts to solve the governing differential equations. Reference 6 contains remarks on many of these approaches. Only two of the relatively new approaches will be discussed here for three reasons:

1. The results being presented herein will be compared with results obtained using both of these previous methods.

2. These methods are both dependent on the moment-thrust-curvature relationship of a particular cross section or group of cross sections as are many of the previous methods. It will be shown in Chapter 3 that the method being presented here is based directly on a material stress-strain curve instead.

3. The other methods are based on either numerical integration or on the solution of the governing equation whereas the method being presented here is based on work-energy principles.

2.2 The Column Deflection Curve (C-D-C), (Refs. 6,12)

The C-D-C method of beam-column analysis is a numerical integration scheme in which a deflected position of a beam column of initially unknown (but estimable) length is found given a set of initial conditions and an appropriate moment-thrust-curvature relationship. Considering for example a simply supported wide flange beam carrying an axial load, P, and a lateral load, Q, the following equations can be written as explained in Ref. 12.

For the first integration step:

$$\delta_{a1} = \theta_o \, \rho_1/2$$

$$M_{a1} = P\delta_{a1} + V_o \, \rho_1/2 - M_o - \text{Applied Moment} \tag{2.1}$$

where M_{a1} = Moment at mid-segment

δ_{a1} = Displacement at mid-segment

M_o = Moment at the first point, i.e., the end of the beam-column.

$$\theta_1 = \theta_o - \phi_{a1} \, \rho_1$$

$$\delta_1 = \theta_o \rho_1 - \tfrac{1}{2} \phi_{a1} \, \rho_1^2$$

$$M_1 = P\delta_1 + V_o \rho_1 - M_o - \text{Applied Moment} \tag{2.2}$$

where θ_o = Initial slope

ρ_1 = Length of the segment

θ_1 = Slope at the end of the segment

δ_1 = Displacement at the end of the segment

M_1 = Moment at the end of the segment

ϕ_{a1} = Average curvature in the first segment

ϕ_{a1} is found from a moment-thrust-curvature relationship with M_{a1} as the average moment in the segment.

Using these values from the first step, the corresponding quantities at the midpoint of the second segment are:

$$\delta_{a2} = \delta_1 + \theta_1 \rho_2/2$$

$$M_{a2} = P\delta_{a2} + V_o(\rho_1 + \rho_2/2) - M_o - \text{Applied Moment} \qquad (2.3)$$

ϕ_{a2} is found from the moment-thrust-curvature diagram so that the displacement, rotation, and moment at the end of the segment can be computed.

$$\theta_2 = \theta_1 - \phi_{a2}\rho_2$$

$$\delta_2 = \delta_1 + \theta_1\rho_2 - \tfrac{1}{2}\phi_{a2}\rho_2^2$$

$$M_2 = P\delta_2 + V_o(\rho_1 + \rho_2) - M_o - \text{Applied Moment} \qquad (2.4)$$

These calculations are performed for increments of lengths, until the centerline slope is zero (for this symmetric example) or until the moment becomes equal to M_{pc}. M_{pc} is the reduced plastic moment corresponding to the applied axial load. Other conditions would apply to different loadings or boundary conditions.

If these calculations are carried out for a sufficient number of combinations of end slope and lateral load for a given axial load then a family of curves relating C-D-C length to end slope for various lateral loads and a given axial load could be drawn. Entering these curves with a given column length would define paired values of end slope and lateral load. Plotting sets of these values for a given length and axial force would produce graphs of lateral load versus end slope from which the maximum value of lateral load could be obtained. This value corresponds to one point on an interaction curve.

2.3 The Column Curvature Curve (C-C-C) (Refs. 1 - 5, 7)

This method of analysis is essentially the solution of the differential equation expressing the equilibrium of the deformed beam-column. In previous attempts at solving this equation deflection was chosen as the independent variable. In the C-C-C method curvature is the independent variable. This leads to lower order differential equation of equilibrium. The behavior of a beam-column of an elastic-plastic material is said to undergo three stages of behavior in which the beam-column is initially elastic, then has plastification on only one side, and finally has plastification on both sides (Ref. 5).

There are then six cases of plastification for a metal beam-column depending on the extent of plastification along and through the beam-column. The maximum strength may occur in any one of the six cases.

In the elastic range the equilibrium equation for a beam-column carrying a uniform load q is:

$$\frac{d^2 m}{dx^2} - P \frac{d^2 y}{dx^2} = -q \tag{2.5}$$

If curvature is defined as

$$\phi = \frac{M}{EI} = - \frac{d^2 y}{dx^2} \tag{2.6}$$

then Eq. 2.5 may be rewritten as Eq. 2.7

$$\frac{d^2 \phi}{dx^2} + K^2 \phi = - \frac{q}{EI}, \quad K^2 = \frac{P}{EI} \tag{2.7}$$

This is a relatively simple differential equation whose solution is:

$$\phi = A \sin kx + B \cos kx - \frac{q}{p} \tag{2.8}$$

The constants of integration are to be found from boundary conditions.

If the equilibrium equation is rewritten as:

$$\frac{d^2 m}{dx^2} + k^2 \phi = 0 \tag{2.9}$$

for the case in which $q = 0$. then this equation is valid in both the elastic and the inelastic cases. If moment can be expressed in terms of curvature and a constant axial force, that relation could be substituted into Eq. 2.9 to produce a differential equation in ϕ. If this equation can be solved then the curvature at any point along the member and for any load level could be found. Reference 1 contains general approximate moment-thrust-curvature expressions in terms of several constants and the resulting differential equations and their solutions. Reference 2 contains guidelines in establishing the constants in the approximate moment-thrust-curvature relationship for more general materials.

The constants resulting from the solution of the differential equations are evaluated using the boundary conditions provided by the six cases of plastification and the "curvature jump" conditions, where applicable. The "curvature jump" conditions arise from the possibility that the derivative of the curvature curve is not unique at some point such as at a concentrated load. Reference 1 contains a discussion of the "curvature jump" condition.

The solutions of the differenetial equations provided by the six plastification cases have been programmed to find corresponding values of lateral load for a value of ϕ and also to find the maximum lateral load (Ref. 7). The points defining a set of interaction curves could be found by varying the length and the axial force for a number of cases and finding the peak lateral load for each case.

The C-C-C method has been extended to reinforced concrete beam-columns (Ref. 4). The tensile strength of the concrete was neglected in that study.

3. THEORETICAL CONSIDERATION

3.1 The Finite Element Method (Refs. 8 - 11, 13, 14)

Consistent with the Finite Element Method the beam-column is divided into elements. Each of the elements is then divided into layers. Elemental and layering discretization are shown in Fig. 1. Each layer of each element may have its own stress-strain curve.

Displacement functions are chosen to describe the displacements within the elements. For a beam element defined by an offset reference axis the following displacement functions were selected (Ref. 8).

$$u = \alpha_1 + \alpha_2$$

$$y = \alpha_3 + \alpha_4 x + \alpha_5 x^2 + \alpha_6 x^3 \tag{3.1}$$

u is the axial displacement and y is the lateral displacement. The constants, α, are determined using the nodal displacements.

$$\{\delta^e\} = [c]\ \{\alpha\} \tag{3.2}$$

$$\{\delta^e\} = \{u_i\ y_i\ \theta_i\ u_k\ y_k\ \theta_k\} \tag{3.3}$$

The generalized stresses and strains are related by the elasticity matrix

$$\{\sigma\} = \begin{Bmatrix} N \\ M \end{Bmatrix} \tag{3.4}$$

$$\{\epsilon\} = \begin{Bmatrix} \frac{du}{dx} \\ \frac{-d^2u}{dx^2} \end{Bmatrix} \tag{3.5}$$

$$\{\sigma\} = [D]\ \{\epsilon\} \tag{3.6}$$

N is the axial force and M is the bending moment in the reference plane. Substitution of the displacement functions, Eq. 3.1, into the definitions of strains, Eq. 3.5, will show that the strains can be related to the constants, α.

$$\{\epsilon\} = [Q]\ \{\alpha\} \tag{3.7}$$

Applying the principle of virtual work as shown in Ref. 14 results in the well known equation for the stiffness matrix. In this case:

$$[K] = [C^{-1}]^T \int_0^{\ell} [Q]^T\ [D]\ [Q]\ dx\ [C^{-1}] \tag{3.8}$$

If plane sections is assumed to be a valid strain distribution the strain in any layer defined as having its centroid a distance Z from the reference axis is given by Eq. 3.9.

$$\epsilon_x = \frac{du}{dx} - Z\frac{d^2y}{dx^2} \tag{3.9}$$

The normal stress, assumed constant throughout the layer, is given by Eq. 3.10

$$\sigma_x = E\,\epsilon_x \tag{3.10}$$

The elasticity matrix [D] can be shown to be given by Eq. 3.11 with the elements defined by Eqs. 3.12.

$$[D] = \begin{bmatrix} \bar{A} & \bar{S} \\ \bar{S} & \bar{I} \end{bmatrix} \begin{Bmatrix} \dfrac{du}{dx} \\ \dfrac{-d^2y}{dx^2} \end{Bmatrix} \tag{3.11}$$

$$\bar{A} = \sum_{i=1}^{J} E_i\,A_i$$

$$\bar{S} = \sum_{i=1}^{J} E_i\,Z_i\,A_i$$

$$\bar{I} = \sum_{i=1}^{J} E_i\,Z_i^2\,A_i + \sum_{i=1}^{J} E_i\,I_{oi} \tag{3.12}$$

where E_i = A layer tangent modulus of elasticity

A_i = A layer area

Z_i = A layer centroidal coordinate

I_{oi} = A layer centroidal moment of inertia

J = The number of layers

Evaluating Eq. 3.8 using Eqs. 3.1, 3.2, 3.3, 3.7, 3.11 and 3.12 yields the element stiffness matrix. In Eq. 3.13, ℓ is the length of the element.

$$[K] = \begin{bmatrix} \bar{A}/\ell & & & & & \\ 0 & 12\bar{I}/\ell^3 & & & \text{Symmetric} & \\ \bar{S}/\ell & -6\bar{I}/\ell^2 & 4\bar{I}/\ell & & & \\ -\bar{A}/\ell & 0 & -\bar{S}/\ell & \bar{A}/\ell & & \\ 0 & -12\bar{I}/\ell^3 & 6\bar{I}/\ell^2 & 0 & 12\bar{I}/\ell^3 & \\ -\bar{S}/\ell & -6\bar{I}/\ell^2 & 2\bar{I}/\ell & \bar{S}/\ell & 6\bar{I}/\ell^2 & 4\bar{I}/\ell \end{bmatrix} \tag{3.13}$$

These equations can be written in incremental form so as to treat a nonlinear problem as a series of piecewise linear problems.

The P-Δ effect caused by the deflection of the beam-column can be included by using the geometric Stiffness Matrix. It has been shown in Ref. 13 that the geometric stiffness matrix relating the axial force to the bending displacement through the second order strain equations is given by Eq. 3.14 in which P is the applied axial force assumed positive if it causes tension.

$$[K_G] = P \begin{bmatrix} 0 & & & & & \\ 0 & 1.2/\ell & & & \text{Symmetric} & \\ 0 & -1/10 & 4\ell/30 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & -1.2/\ell & 1/10 & 0 & 1.2/\ell & \\ 0 & -1/10 & -\ell/30 & 0 & 1/10 & 4\ell/30 \end{bmatrix} \tag{3.14}$$

Combining Eqs. 3.14 and 3.14 gives the equilibrium equation for the displaced beam-column element.

$$\{F\} = \Big[[K] + [K_C] \Big] \{\delta\} \tag{3.15}$$

The stiffness matrices of each element can then be assembled to form the global equilibrium equations. After application of the boundary conditions these equations can be solved for each increment of load.

3.2 Stress-Strain Curves (Refs. 8,9)

In order to have an essentially material independent computer program the Ramberg-Osgood law, Eq. 3.16 below, was chosen as the basis for stress-strain curves.

$$\epsilon = \frac{\sigma}{E} + \left(\frac{1 - \bar{M}}{\bar{M}}\right)\left(\frac{\sigma_1}{E}\right)\left(\frac{\sigma}{\sigma_1}\right)^{\bar{N}} \tag{3.16}$$

where ϵ = Strain

σ = Stress

E = An initial modulus of elasticity

σ_1 = A secant yield strength

$\bar{N}$ = A constant

$\bar{M}$ = A constant defining a slope of $\bar{M}.E$ on a stress-strain curve

The application of this curve to metals is well established. With $\bar{M} = 0.7$ and $\bar{N} = 100.0$ or more, a satisfactory approximation to an elastic-perfectly plastic stress-strain curve seems to be obtained (Ref. 8).

Reference 8 contains a detailed description of the application of the Ramberg-Osgood curve to concrete and contains a comparison of the results obtained with other stress-strain curves for concrete. The following information is a summary of that presentation.

3.2.1 Concrete in compression

a) Find E from any acceptable equation.

b) Define $\bar{N} = 9$, $\sigma_1 = f'_c$, $\bar{M} = f'_c/0.002E$

c) Use the resulting Ramberg-Osgood law to a strain of 0.002

d) Provide a horizontal plateau from a strain of 0.002 to ϵ_1 which is dependent on f'_c.

e) Provide a downward leg at a slope of E_D which is also dependent on f'_c.

Reference 8 contains a table of suggested values for E_D and ϵ_1.

3.2.2 Concrete in Tension

a) A Ramberg-Osgood curve has been provided up to a stress equal to a maximum tensile stress.

b) Two downward line segments have been provided which require two slopes and a strain at which the slope changes.

This tensile curve has been provided to allow for future developments in tensile stress-strain curves. Until more information is known the following two line segment curve is recommended. Reference 8 has shown it to give good agreement between analytic and experimental load-deflection diagrams of two reinforced and eleven prestressed concrete beams.

a) Assume a loading curve which is a straight line of slope equal to the compressive modulus of elasticity and terminating at the maximum tensile stress.

Setting $M = 1.0$, $\sigma_1 = F_t$ in the Ramberg-Osgood curve with $\sigma \leq F_t$ will provide this straight line.

b) Assume an unloading curve which is linear downward at a slope of 800.0 ksi.

Strain hardening can also be handled by supplying a negative slope on the downward legs used in the tensile and compressive stress-strain curves. Reference 9 contains a detailed description of the options allowed by the original computer program.

The downward legs of the stress-strain curves are used to convert strain increments into "ficticious stresses" which are in turn used to unload layers which have been found to exceed cracking or crushing criteria. The "ficticious stresses" are also used to compute nodal forces which hold the rest of the beam-column in equilibrium. This process produces a globally adequate but not locally exact redistribution of stresses.

Each layer may have its own stress-strain curve. In this way non-homogeneous beam-columns can be analyzed. The assumption of plane sections implies, of course, that no relative slip between materials of a given beam-column may occur. When dealing with concrete members this means that perfect bond has been assumed.

3.3 Iteration Scheme (Refs. 8,9)

The iteration procedure for a given load increment is started by solving the global equilibrium equations for the increments of displacement. Strain increments are computed from the displacement increments. Using the latest level of stress available new tangent moduli are computed for each layer, the global stiffness matrix is regenerated and the equilibrium equations are solved again. If the new increments of displacement are within a relative tolerance of the previous set, convergence is said to have occurred. If convergence has not occurred the process is repeated. If convergence is not attained in several trials the load increment is reduced and the process is repeated. If no convergence is attained after a number of reductions in load the process is stopped. If convergence is attained in relatively few trials the load increment to be applied for the next load step is increased so as to reduce the total number of load steps used.

Once convergence has been attained for the load step, consideration is given to cracking and crushing if appropriate. The first phase in this step is a pre-scanning process in which all the layers are checked to see if they have exceeded the allowable tensile or compressive stress tolerances by an excessive amount. If this occurs the basic load step is reduced and the problem of finding a converged displacement increment for the basic load step is repeated.

Once it has been determined that no stress criteria are exceeded by more than their tolerances any alteration in stiffness required by the cracking or crushing of a layer is made. The "ficticious forces" described in Section 3.2 are computed.

The global equilibrium problem corresponding to that set of "ficticious forces is solved until convergence is attained. The layers are then rechecked to see if subsequent cracking or crushing has occurred. If so the cracking-crushing analysis is repeated. It is the process of cracking generating more cracking and/or crushing generating more crushing which simulates the in-plane instability condition in concrete beam columns.

Alterations in the stiffness matrix arising from plastic flow like phenomena are automatically accounted for by employing the appropriate Ramberg-Osgood curve.

As originally coded the computer program computed the basic load step load vector from lateral loads only. As such, the incremental iterative process would be performed only on the lateral load applied to the beam-column. It would be a relatively simple matter to include a nodal load vector in the incremental load vector and thus allow for the case of proportional loading involving an eccentric (or concentric) axial force and a lateral load.. This was not actually done because the comparisons with previous work involved solutions based on the application of a constant axial force first and then the application of an increasing lateral load. One of the advantages of the Finite Element Method is that there is nothing conceptually prohibitive about incrementing all the loads applied to the beam-column. This will be proposed as a possible area of future research.

Another worthwhile change in the iteration procedure would be to allow for two separate iterated load vectors. This would be especially helpful for more extensive work with concrete beam-columns in which the application of a large axial force may, by itself, cause substantial nonlinear behavior to occur before the lateral load is applied. This problem arose in two sets of comparitive examples to be presented in Chapter 4. Conversion to two incremented load vecotrs was considered beyond the scope of a pilot study but would be a most desirable addition to the program if an extensive study of beam-columns were to be performed.

4. NUMERICAL RESULTS

4.1 Introductory Comments

Figure 2 shows load-deflection curves for an 8 x 40 wide flange shape with an L/r ratio of 47.3. The span was 14 feet with third point loading. This is the only figure based on an 8 x 40 section and is presented to make fundamental observations on characteristics of the method being reported. Comparisons with existing solutions for steel wide flange beams will be based on the 8 x 31 section.

Figure 2 shows curves for $P/P_y = 0.$, $P/P_y = .3$ without including the geometric stillness matrix, $P/P_y = .3$ including the geometric stiffness matrix and $P/P_y = .6$. The following observations are made:

1. Increasing P/P_y decreases M_{pc}, as it should.
2. Increasing P/P_y without including the geometric stiffness matrix would overestimate the stiffness of the beam-column and overestimate the ultimate load.
3. Increasing P/P_y while including the geometric stiffness matrix results in a more flexible system.

A more important observation deals with the behavior occurring at the peak of each curve. Those curves for steel sections without the geometric stiffness matrix will have steadily increasing deflections due to the formation of a plastic hinge. Those curves which include the geometric stiffness have reached a point where the combined stiffness matrix $[K + K_G]$ is no longer positive definite for the next load increment. Thus the failure of this type of example is due to loss of stability. It will be seen in the discussion of concrete beam-columns that they can fail by loss of stability due to beam-column action or by the cracking and crushing caused by excessive bending.

The curves shown in Fig. 2 are not as "smooth" in the nonlinear range as the rest of the load deflection curves for the 8 x 31 shape. This is a result of the type of layering used.

4.2 Steel Wide Flange Beam-Columns

Figure 3 shows a comparison of interaction curves for a concentric axial load reported by Lu and Kamalvand (Ref. 12) using the column deflection curve method, Chen (Ref. 2) using the column curvature method and the results of this study shown by the dashed lines. Lu and Kamalvand used a yield stress of 36 ksi whereas Chen and this study used 34 ksi. Accordingly, the column deflection curve results for L/r = 80 have to be adjusted as described in Ref. 12.

$$(L/r)_{36} = (L/r)_{34} \sqrt{36/34}$$

The results for L/r = 20 were not adjusted because the change was too small to effect a figure of this scale.

The general agreement appears to be quite good. The three points showing the largest apparent discrepancy compared with the column curvature results are ($L/r = 20$, $P/P_y = 0.2$), ($L/r = 20.$, $P/P_y = 0.3$) and ($L/r = 80.$, $P/P_y = 0.85$). In each case the stress field output of these examples showed a premature failure to converge in the next load step beyond what is plotted. In the discussion of iteration procedures in Section 3.3 it was noted that load reduction was used to attain convergence in the original program based on some number of unsuccessful trials. In the case of the three points being discussed here better results would probably be obtained if the load step was reduced when apparent failure to converge developed due to deterioration in the condition of the stiffness matrix. This would still develop at a slightly higher load but such a reduction might allow one or two more load steps to be taken in some cases.

The elemental and layering discretizations are shown in Fig. 1. Previous results of a parametric study using the original computer program indicate that the number and location of the elements and layers used here are more than adequate for this analysis.

Figures 4 through 6 show load deflection curves for various P/P_y ratios and a given L/r ratio. These figures show the substantial effect that increasing the axial load has on the ability to carry the lateral load. They also show the effect that increasing the axial load has on the stiffness of the beam column.

Increasing the axial load decreases the stiffness. It is again noted that without the geometric stiffness matrix all of the curves on each of Figs. 4, 5, and 6 would have the same slope in the elastic range. Comparing Figs. 4, 5 and 6 shows that the effect of the geometric stiffness matrix increases with increasing length.

Figure 7 shows interaction curves for the same beam-columns with an eccentric axial load. Comparisons are made only with the column curvature results by Chen (Ref. 2). The agreement between the column curvature and the finite element results are generally even better than those presented for the concentric axial load.

The load-deflection curves are shown in Figs. 8, 9 and 10. The only new observation here is that the curves for $P/P_y \neq 0$. appear to have a change in slope in the elastic portion of the curve. This is a result of the simultaneous application of the eccentric axial force and initial lateral load. An estimate of the centerline deflection caused by the eccentric axial force acting alone can be made by extrapolation as shown in the figures.

4.3 Reinforced Concrete Beam-Columns

The rectangular, doubly reinforced section used by Chen and Chen (Ref. 4) was also used here. The cross section is shown in Fig. 1. The beam-column is 14 inches deep, 12 inches wide and has equal compressive and tensile steel areas totaling 6.72 square inches. The nominal compressive strength was 3.0 ksi but Chen and Chen used Hognestad's stress-strain curve to define their moment-thrust-curvature relations. This stress-strain curve assumes a 15% reduction in nominal compressive strength as the peak beam-column compressive stress. Accordingly a value of $\sigma_1 = 2.55$ ksi was chosen to approximate the reduction in the Ramberg-Osgood curve. Chen and Chen neglected the tensile strength of the concrete whereas the method being reported includes it.

The resulting interaction curve for the concentric axial load case is shown in Fig. 11. It can be seen that the agreement between the results of both analyses agree quite well for the curves with L/t = 30. and L/t = 20. The agreement with the previously reported results for L/t = 10. is not as striking but is still within about 5% of the same value of Q/Q_o for a given value of P/P_o. The differences in the stress-stain curves used in both approaches may account for some of the differences.

The corresponding load-deflection curves are shown in Figs. 12, 13 and 14. It will be noted that these curves do not appear as systematic as those presented for the examples using steel sections. There are two reasons for this:

1. As shown in Fig. 11 it is possible for the beam-column to support a larger lateral load for some values of axial load than it can without the axial load.

2. The effect of cracking is evident in these load-deflection curves. It appears as a relatively early change in slope of the load deflection curve. The amount of change in slope depends on the extend of the spread of cracked layers along and into the beam-column.

It will be noticed in each of Figs. 12, 13 and 14 that before cracking becomes evident the previously noted decrease in stiffness with increasing axial load is still apparent. Thus the effect of the geometric stiffness matrix is also seen in the load-deflection behavior of concrete beam-columns.

Figure 15 is the interaction diagram for an eccentically load reinforced concrete beam-column. Good agreement with the work of Chen and Chen is found. The finite element results do not extend as far along these interaction curves because of the limitations in the iterative procedure. For the higher values of P/P_o for both the eccentric and the concentric case the axial load alone caused enough nonlinear behavior to result in failure to converge to the first displacement increment. This first displacement increment had to correspond to the entire axial load because, as explained in Section 3.3, there were no provisions to increment the axial load. For the concentric load case it was relatively easy to circumvent this problem by using an initial stress field which satisfied equilibrium and strain compatability. Strain criteria were adjusted accordingly. No such simple expedient was tried with the eccentric beam-columns because each layer would have to have its own stress-strain curve in order to accommodate the change in strain criteria. It was felt that the results presented in Figs. 3 through 18 were conclusive enough without the added evidence obtained by one or two more points on Fig. 15.

Figures 16, 17 and 18 show the load-deflection curves corresponding to Fig. 15. These curves have the offset previously noted in Figs. 8, 9 and 10. Their shape is intrinsically different from Figs. 12, 13 and 14 because of the bending moment provided by eccentric axial load. This is also evident in a comparison of Figs. 11 and 15.

4.4 Increasing Solution Efficiency

The incremental iterative method described here is relatively efficient compared to similar techniques which have been used to analyze beams using a continuum approach. This is a result of the relatively few equilibrium equations which have to be solved. Reference 8 contains a detailed discussion of this point. However, compared to some other beam-column analysis techniques this method requires a large computational effort. This assumes, of course, that another solution exists for a given problem. There are two relatively obvious ways to reduce computational effort even more.

The first way to increase efficiency would be to eliminate the entire cracking process by assuming as Chen and Chen did

(Ref. 4) that concrete has no strength in tension. To test this possibility the reinforced concrete beam column of Section 4.3 with an L/t = 10. was solved using a tensile strength of only 20. psi. Using a value this low would simulate the effect of zero tensile strength.

The second way to increase efficiency would be to reduce the number of equilibrium equations still further by using fewer elements. A parametric study (Ref. 11) on inelastic beams using the original computer program indicated that reliable load-deflection curves could be obtained for beams with fewer elements provided care was used in discretization. It would therefore be logical to expect that good interaction curves could be obtained by using fewer elements. To investigate this possibility the examples with a tensile strength of 20 psi were also investigated simulating the analysis which would result from using six elements. This was done by using four elements each only 0.05% of the length of the beam-column.

The results of this analysis is shown in the table below in ratio form. The base value is taken as the ten element model with a tensile strength of 400 psi.

No. of Elements	Tensile Strength	Lateral Load Ratio for P/P_o = 0.2	0.4	0.6	0.8
10	400	1.000	1.000	1.000	1.000
10	20	0.995	0.995	0.993	1.000
6	20	1.007	0.995	1.002	1.004

As can be seen from the table, each of these analyses would give virtual identical points on an interaction curve. It is conservatively estimated that implementation of both the reduction in the number of elements and neglecting the tensile strength of the concrete could reduce solution time by 50% or more.

Figures 19 through 23 are load-deflection curves for the original and both modified reinforced concrete beam-column models. It can be seen that neglecting the tensile strength of the concrete had relatively little effect on the shape of the load-deflection curve, especially for $P/P_o \geq 0.4$. This is a result of the examples with $P/P_o \geq 0.4$ being in the compression failure zone of the interaction diagram, Fig. 11. The example with $P/P_o = 0.2$ is in a transition region. For $P/P_o < 0.2$ the effect of neglecting the tensile strength would be more pronounced in the load deflection curve. This is seen in Fig. 19 for which $P/P_o = 0.0$. As would be expected from Fig. 11, the effect of neglecting the tensile strength is reduced as P/P_o is increased. This is because less and less of the beam column is in tension. In fact when $P/P_o = 0.8$ there was no tension at all.

5. CONCLUSION AND FUTURE RESEARCH

It can be concluded from this research that the incremental iterative Finite Element Method using a simple layered beam element can provide solutions to inelastic beam-column problems. As previously noted in Chapters 1 and 2 there is already a large body of inofrmation in this area. The method used here is a relatively laborious procedure compared to other existing methods assuming that they have been applied to a given problem. It does, however, have several advantages which might prove useful in future beam-column studies especially if the changes to the iteration scheme mentioned in Chapters 3 and 4 are made. These advantages are:

1. A wide range of loadings can be handled. There is no intrinsic difference between one concentrated load, several concentrated loads, uniform loads,symmetric loads or unsymmetric loads.

2. Boundary conditions can also be handled easily. There is no change in the formulation required for different boundary conditions.

3. There is no need for an a-priori moment-thrust-curvature curve.

4. There is nothing conceptually prohibitive about changing the order of the loading or using simultaneous (but proportional) axial and lateral loads.

5. The previous work with prestressed concrete beams (Ref. 8) would indicate that prestressed concrete beam-columns could also be treated.

This method might also be used as a check on future extensions of beam-column analysis techniques such as column deflection curves or column curvature curves. As such it might provide an independent solution such as seen in Figs. 3, 7, 11 and 15. It is again noted that the changes already mentioned should be performed before more extensive beam-column studies are conducted.

6. FIGURES

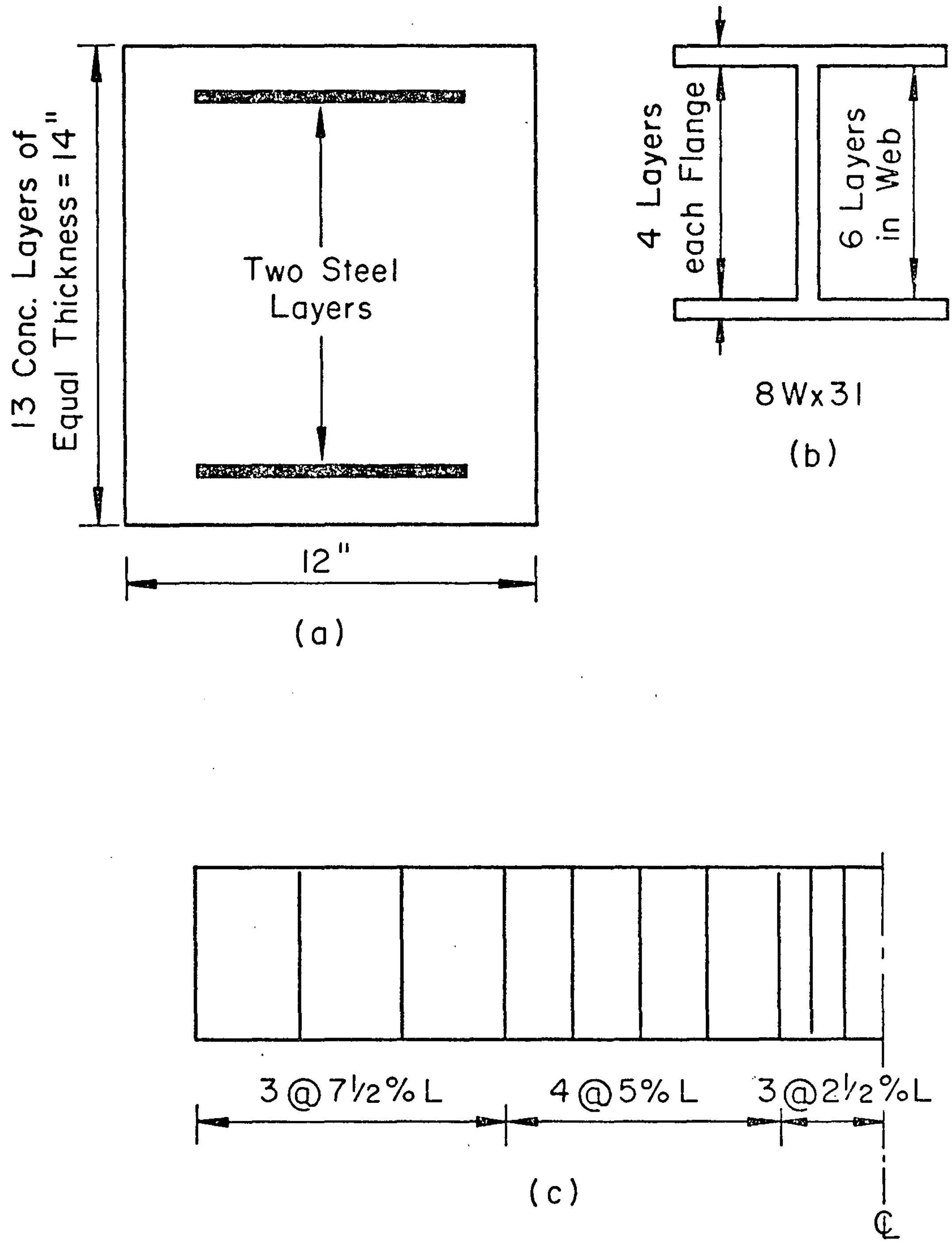

Fig. 1 Discretization Models

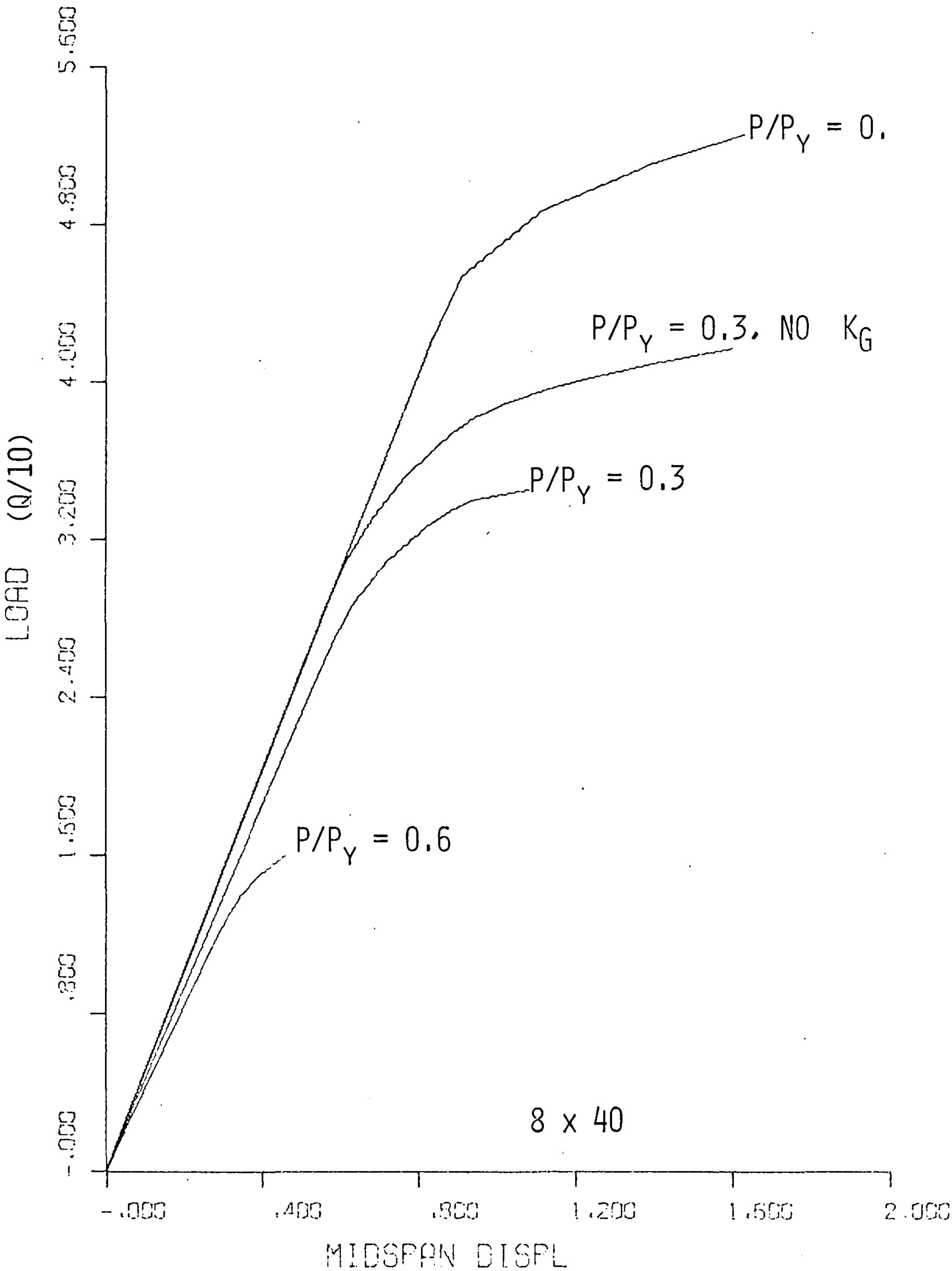

Fig. 2 Load Deflection Curve for 8 x 40

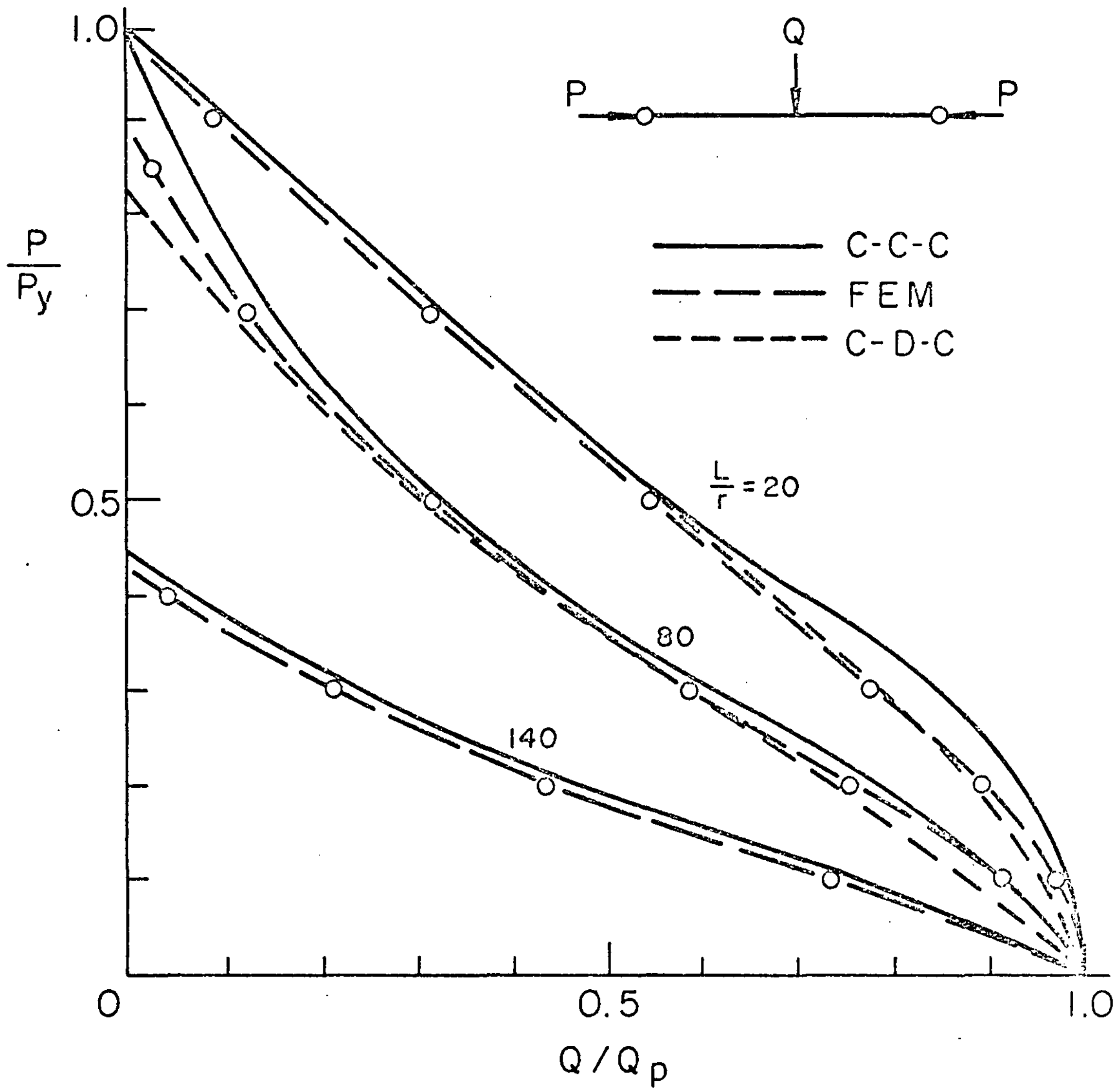

Fig. 3 Interaction Curve for 8 x 31

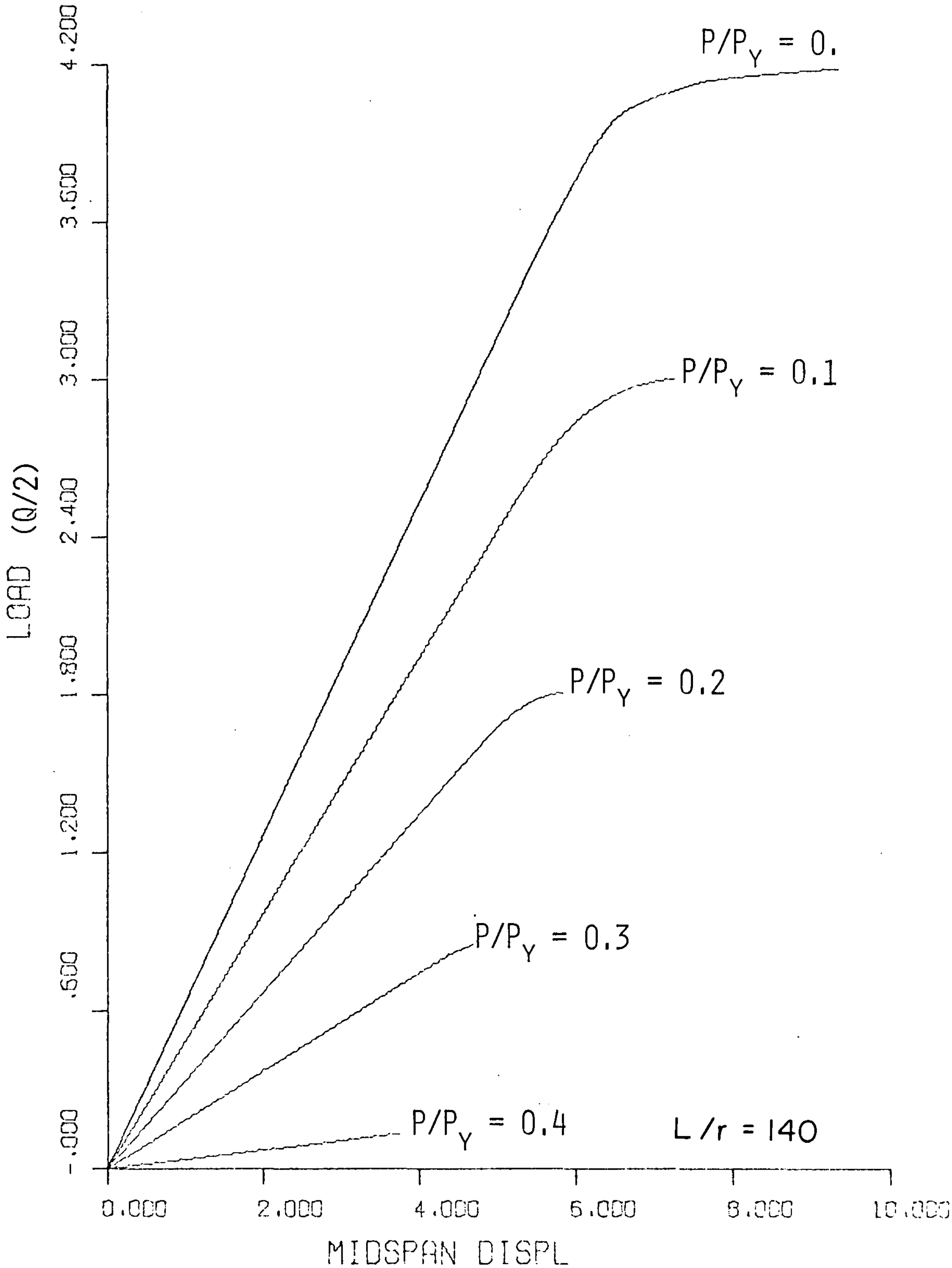

Fig. 4 Load Deflection Curves for 8 x 31

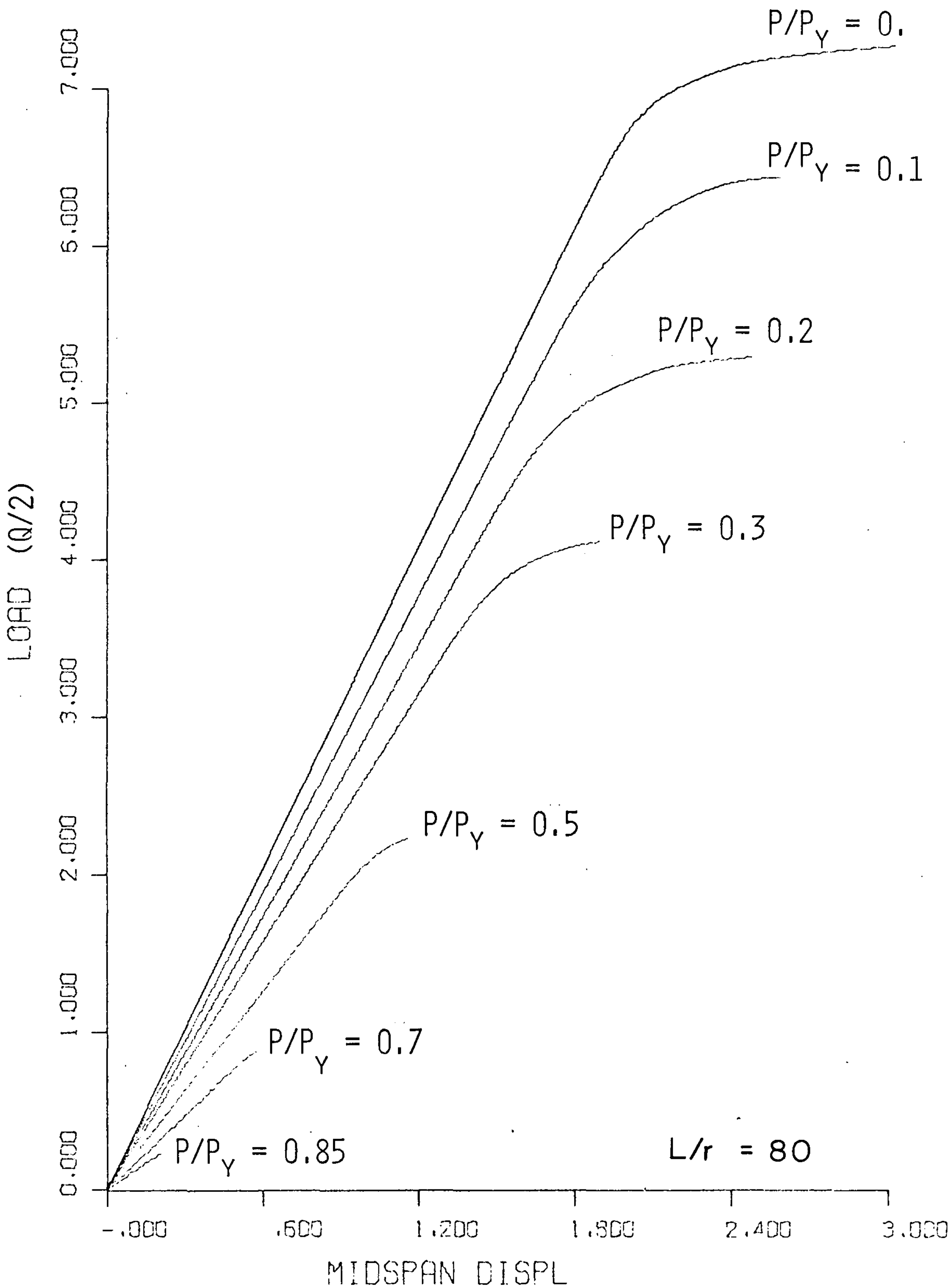

Fig. 5 Load Deflection Curves for 8 x 31

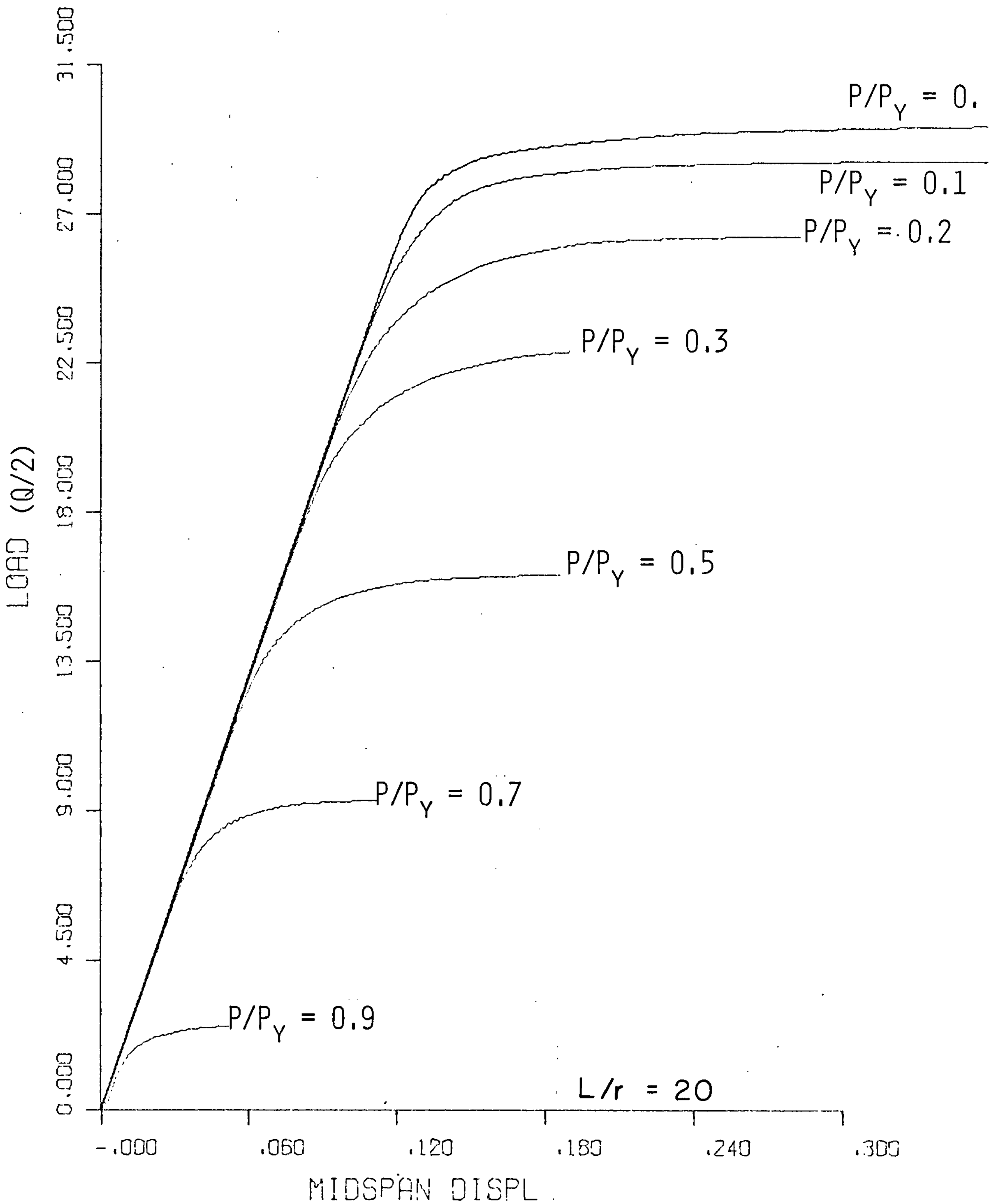

Fig. 6 Load Deflection Curves for 8 x 31

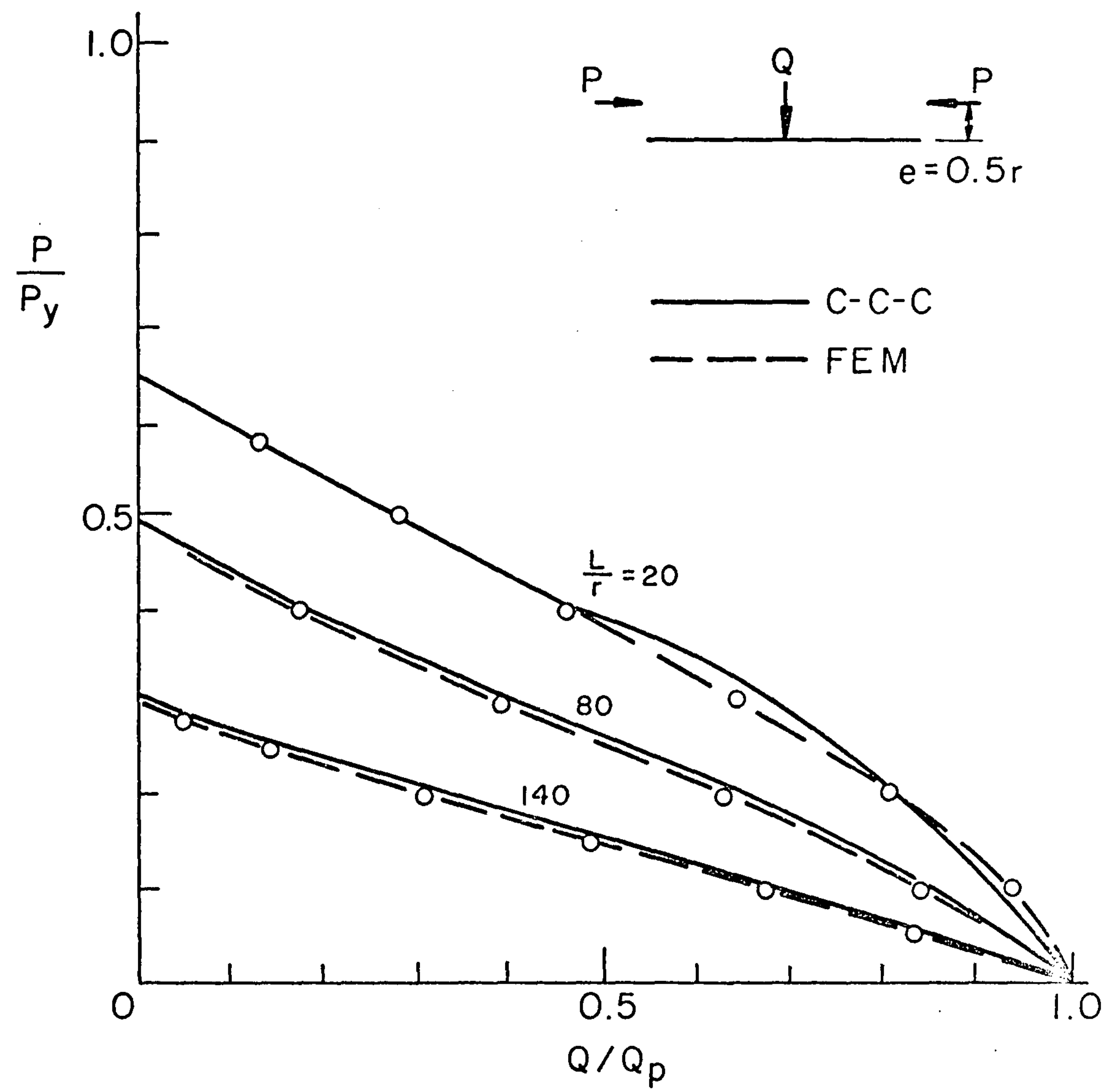

Fig. 7 Interaction Curve for 8 x 31

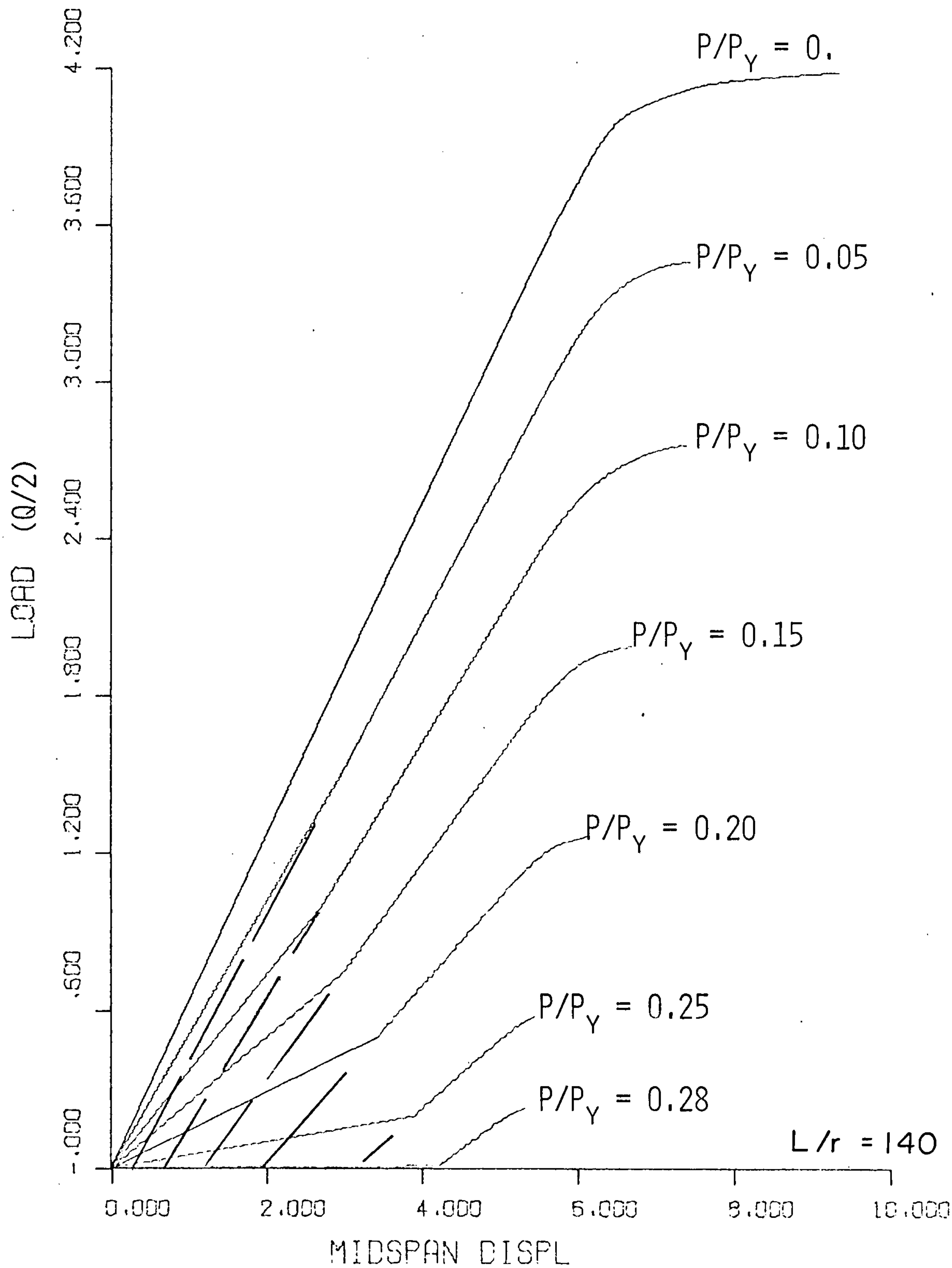

Fig. 8 Load Deflection Curves for 8 x 31

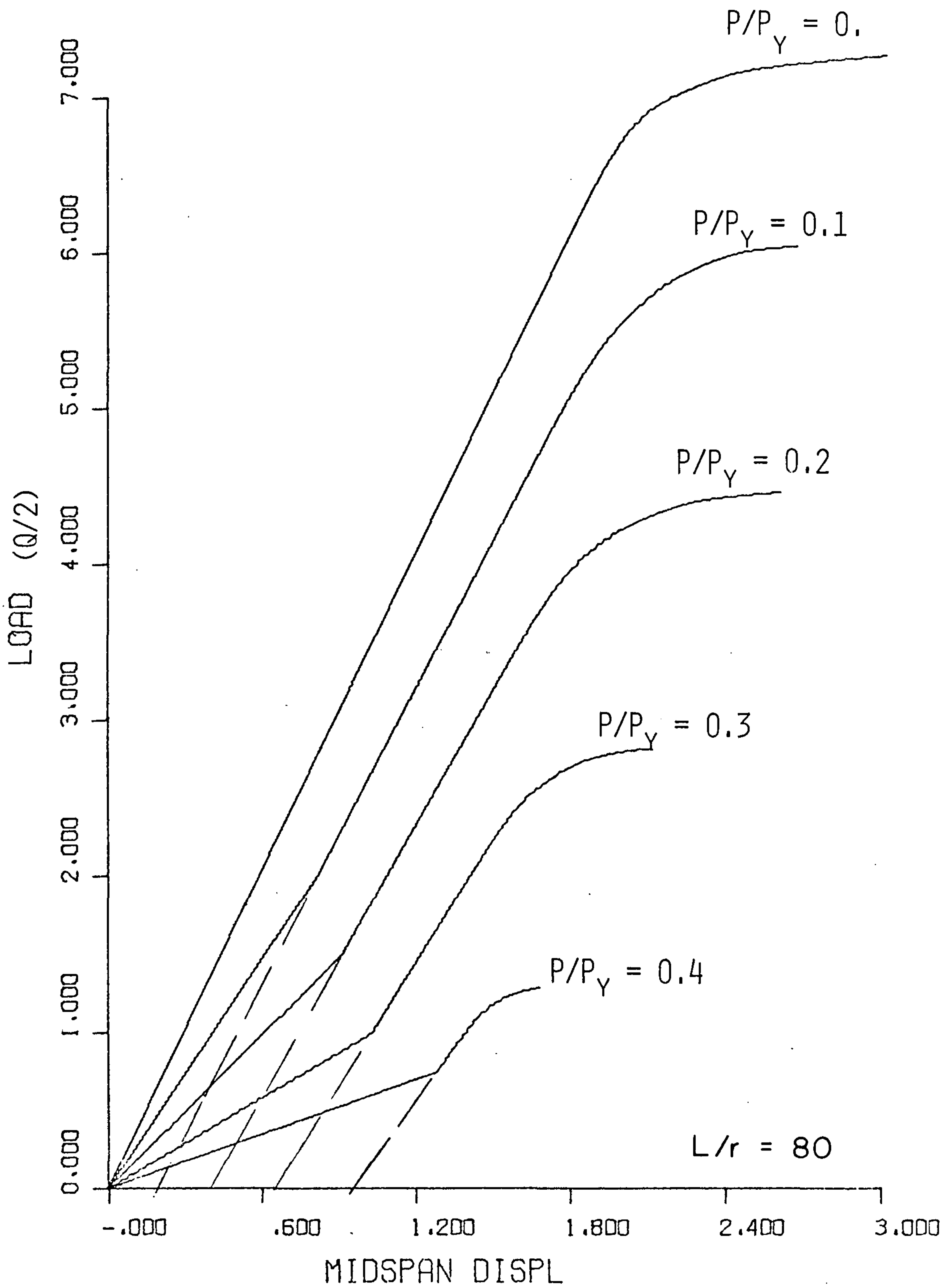

Fig. 9 Load Deflection Curves for 8 x 31

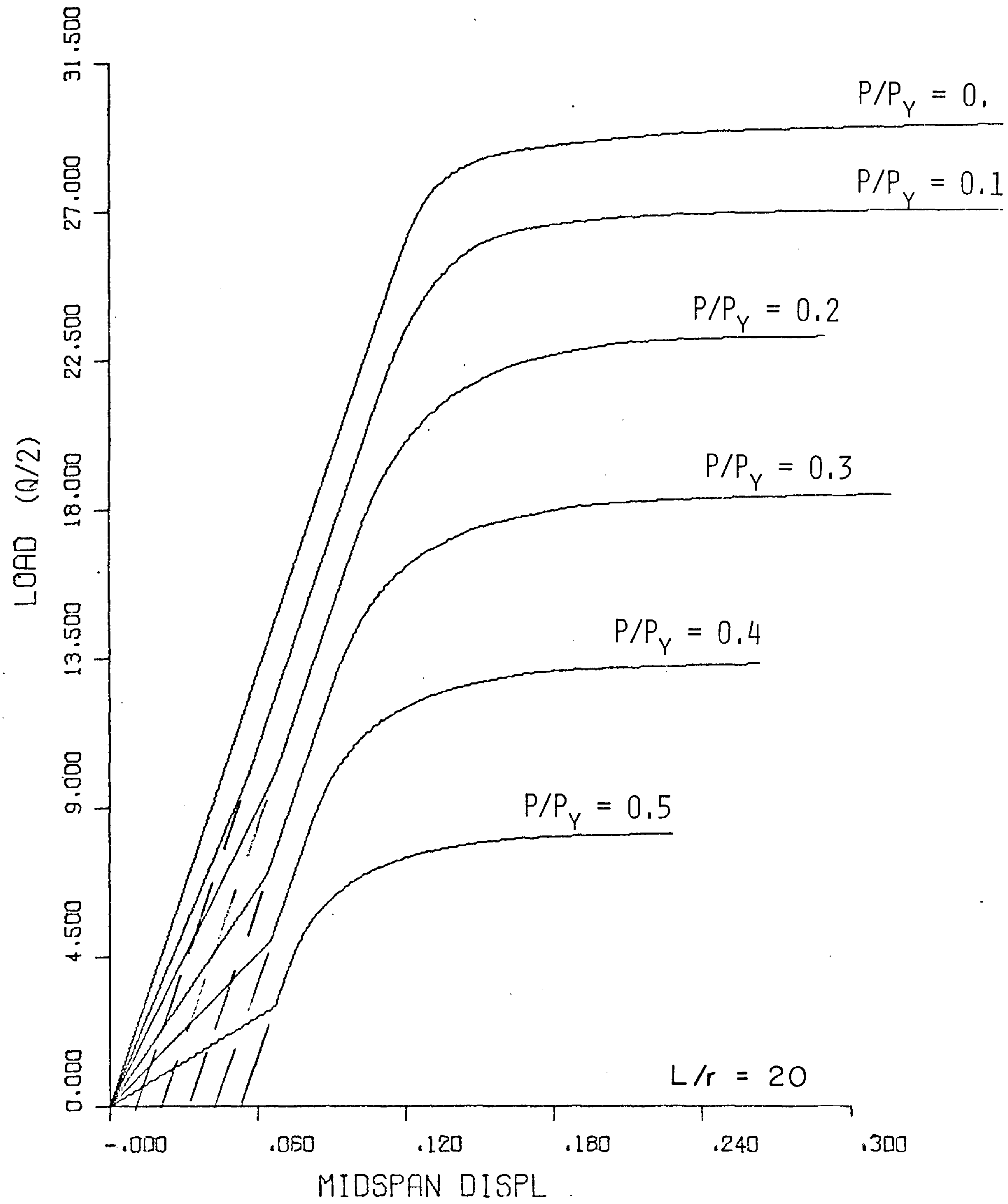

Fig. 10 Load Deflection Curves for 8 x 31

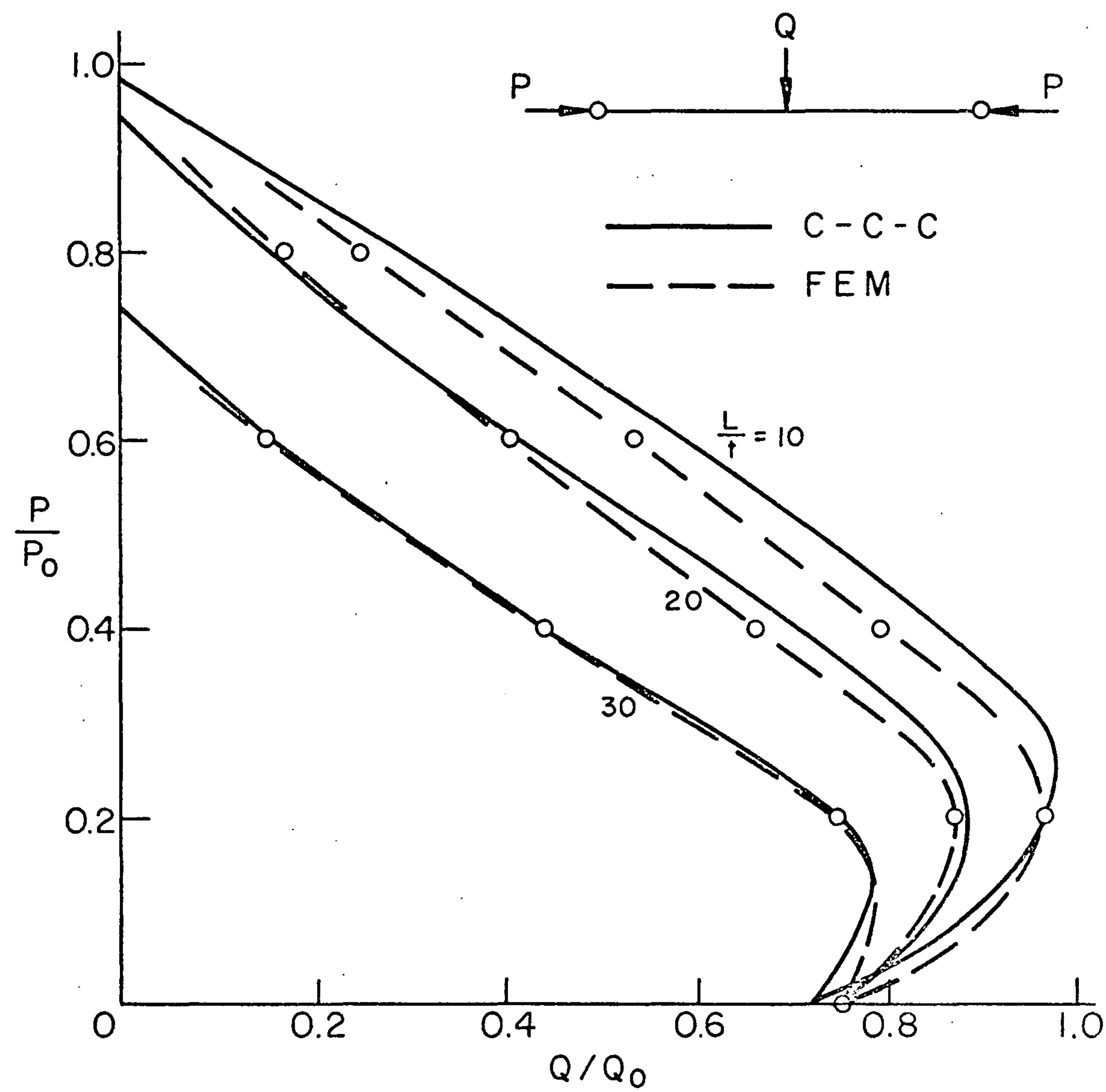

Fig. 11 Interaction Curves for Reinforced Concrete Beam Column

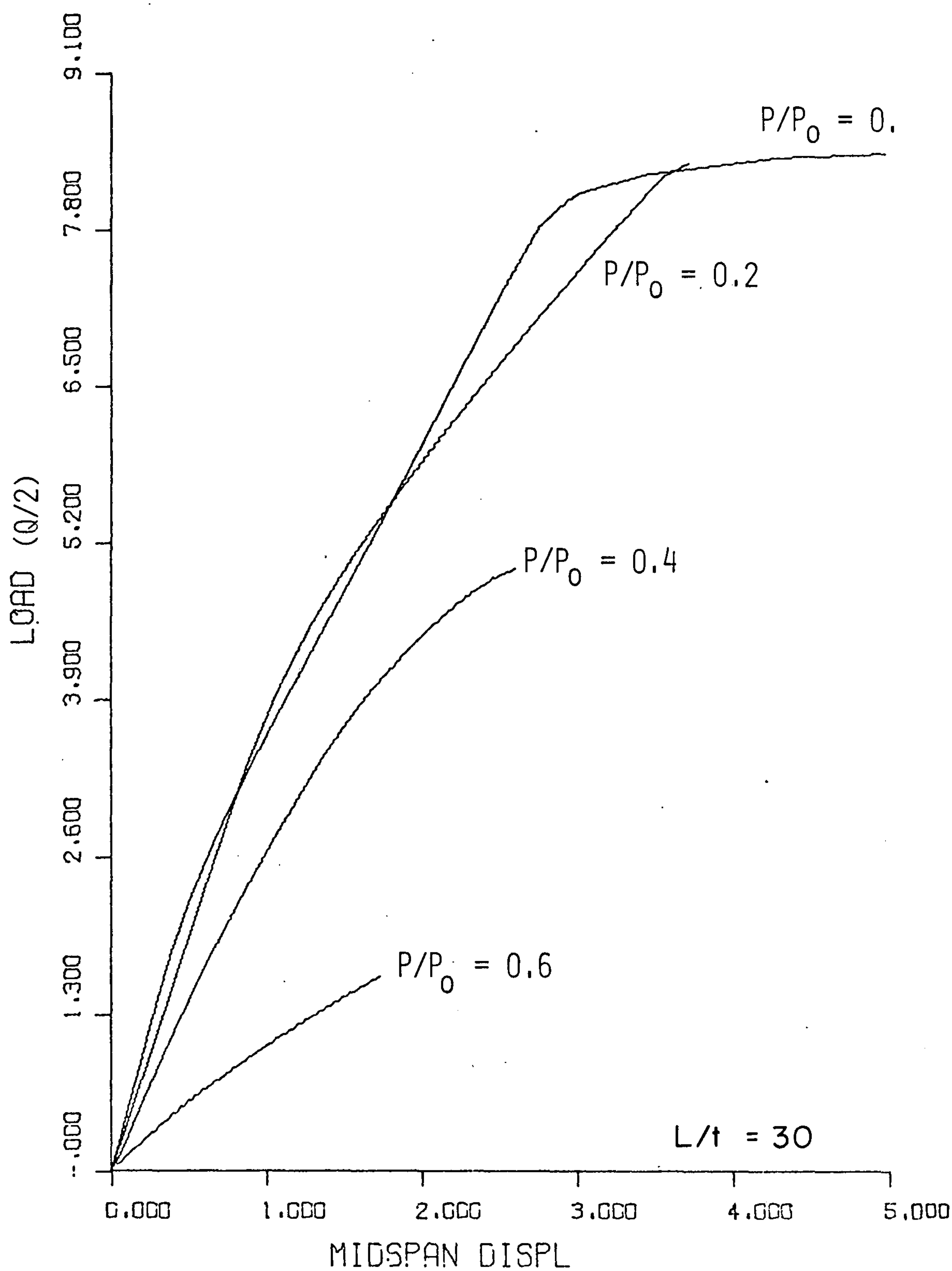

Fig. 12 Load Deflection Curves for Reinforced Concrete Beam Column

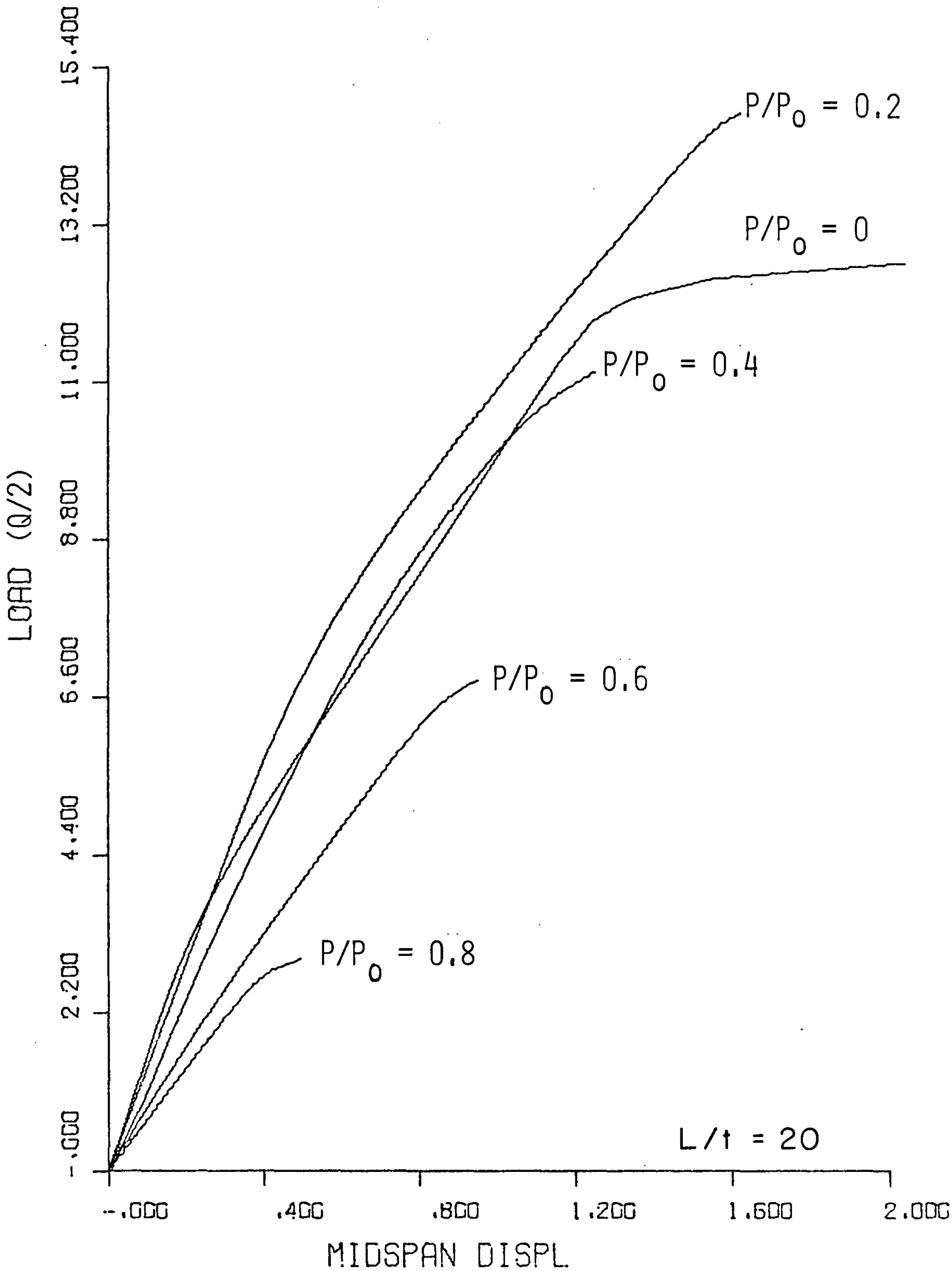

Fig. 13 Load Deflection Curves for Reinforced Concrete Beam Column

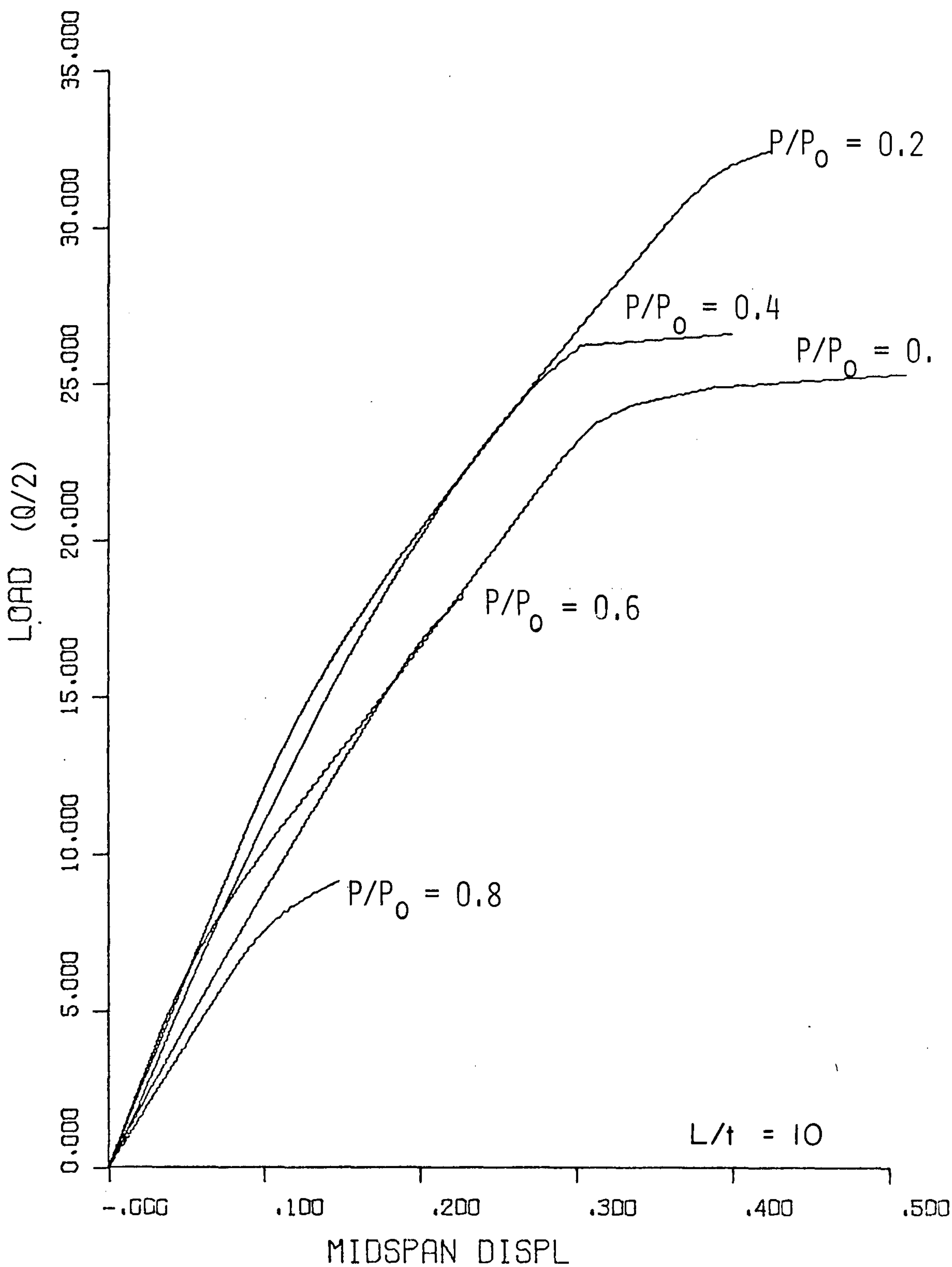

Fig. 14 Load Deflection Curves for Reinforced Concrete Beam Column

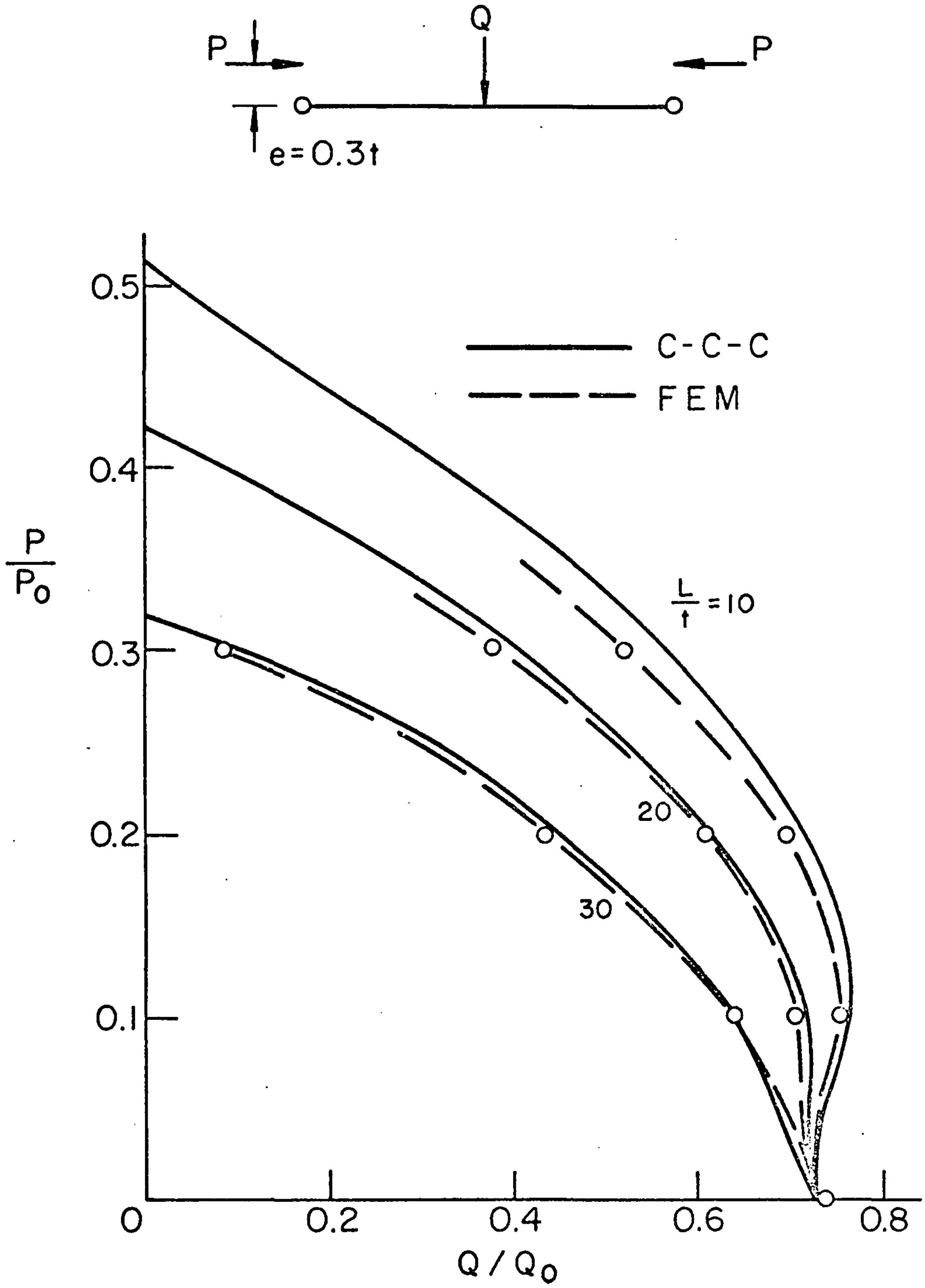

Fig. 15 Interaction Curves for Reinforced Concrete Beam Column

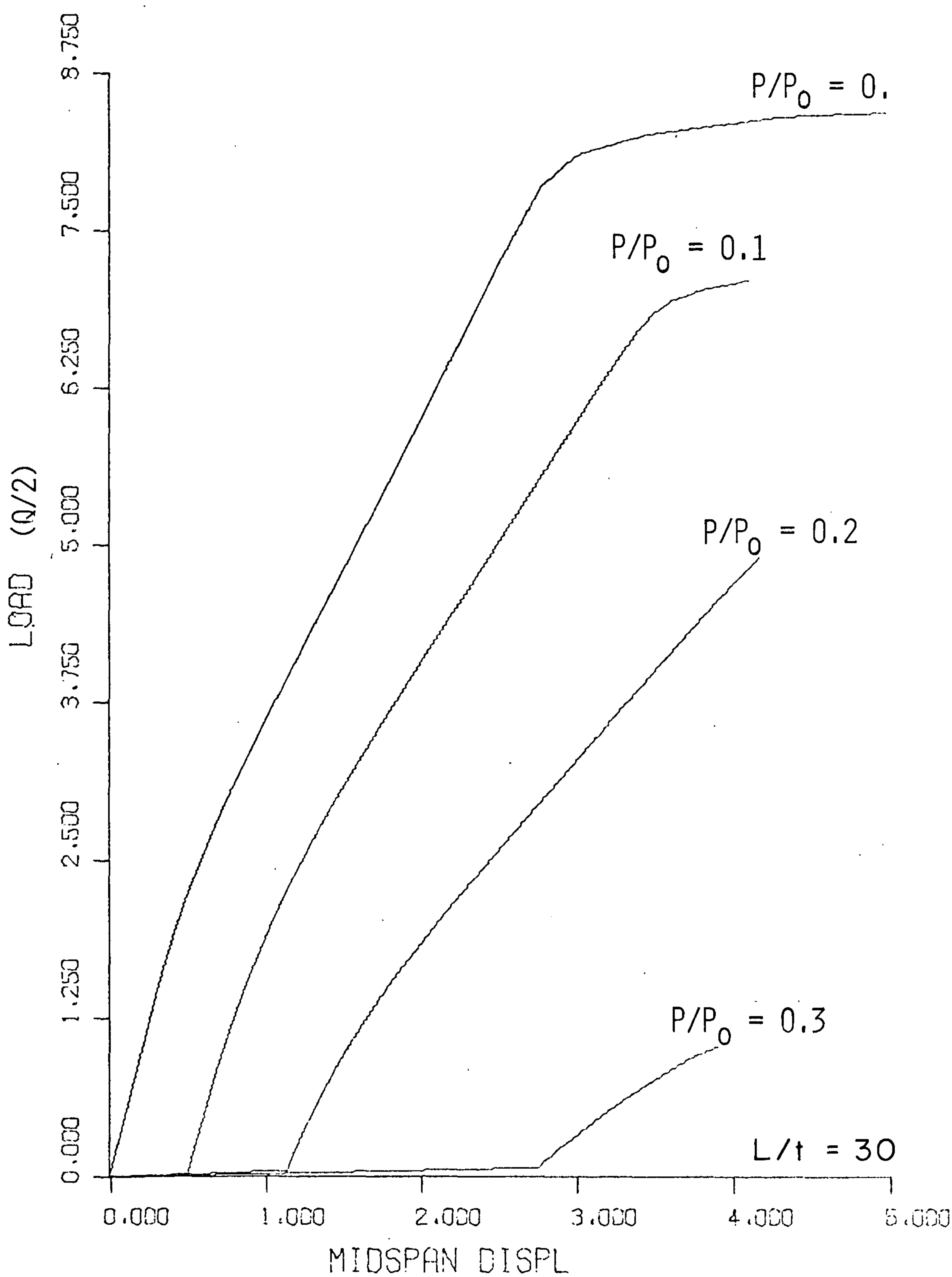

Fig. 16 Load Deflection Curves for Reinforced Concrete Beam Column

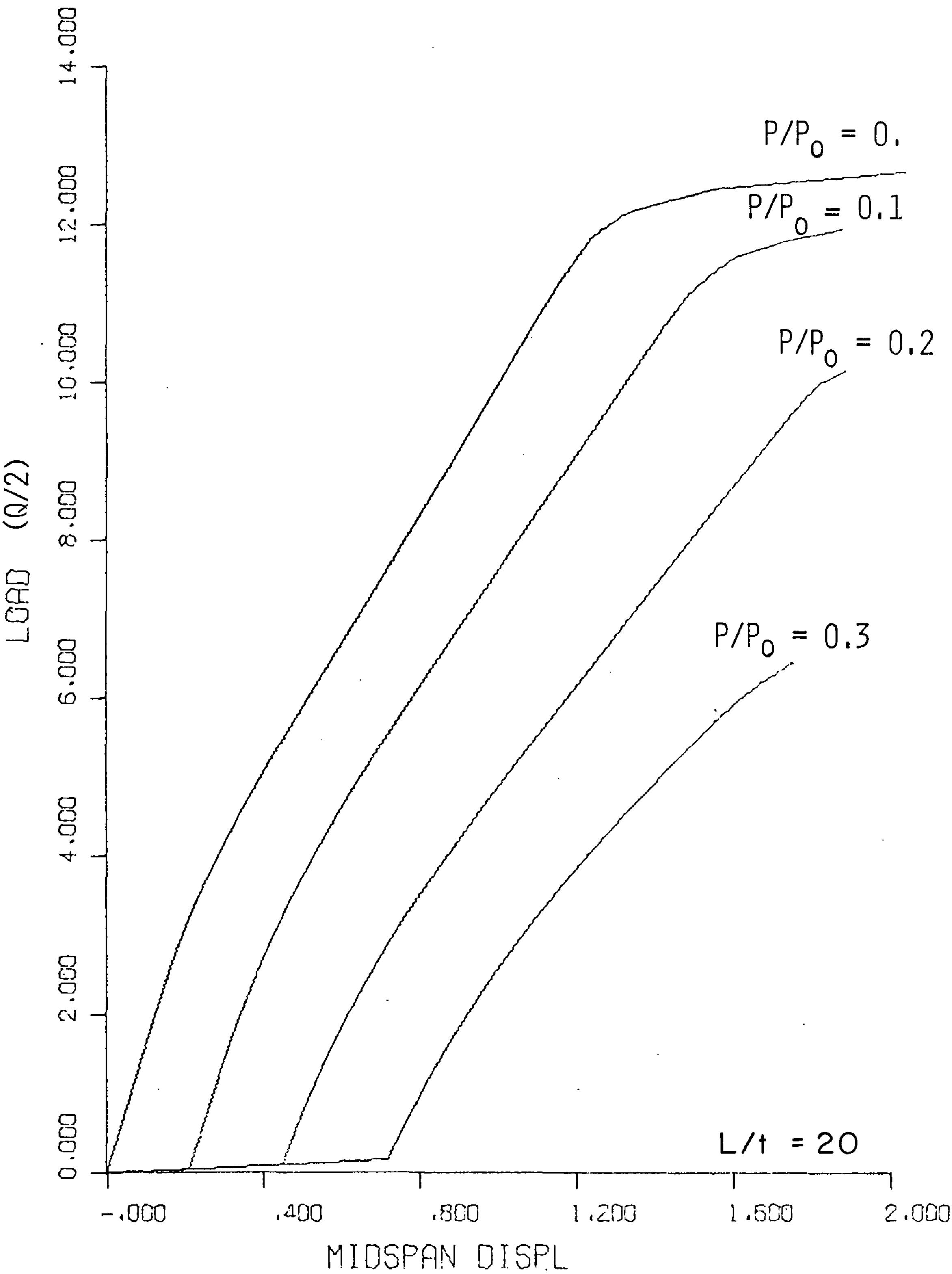

Fig. 17 Load Deflection Curves for Reinforced Concrete Beam Column

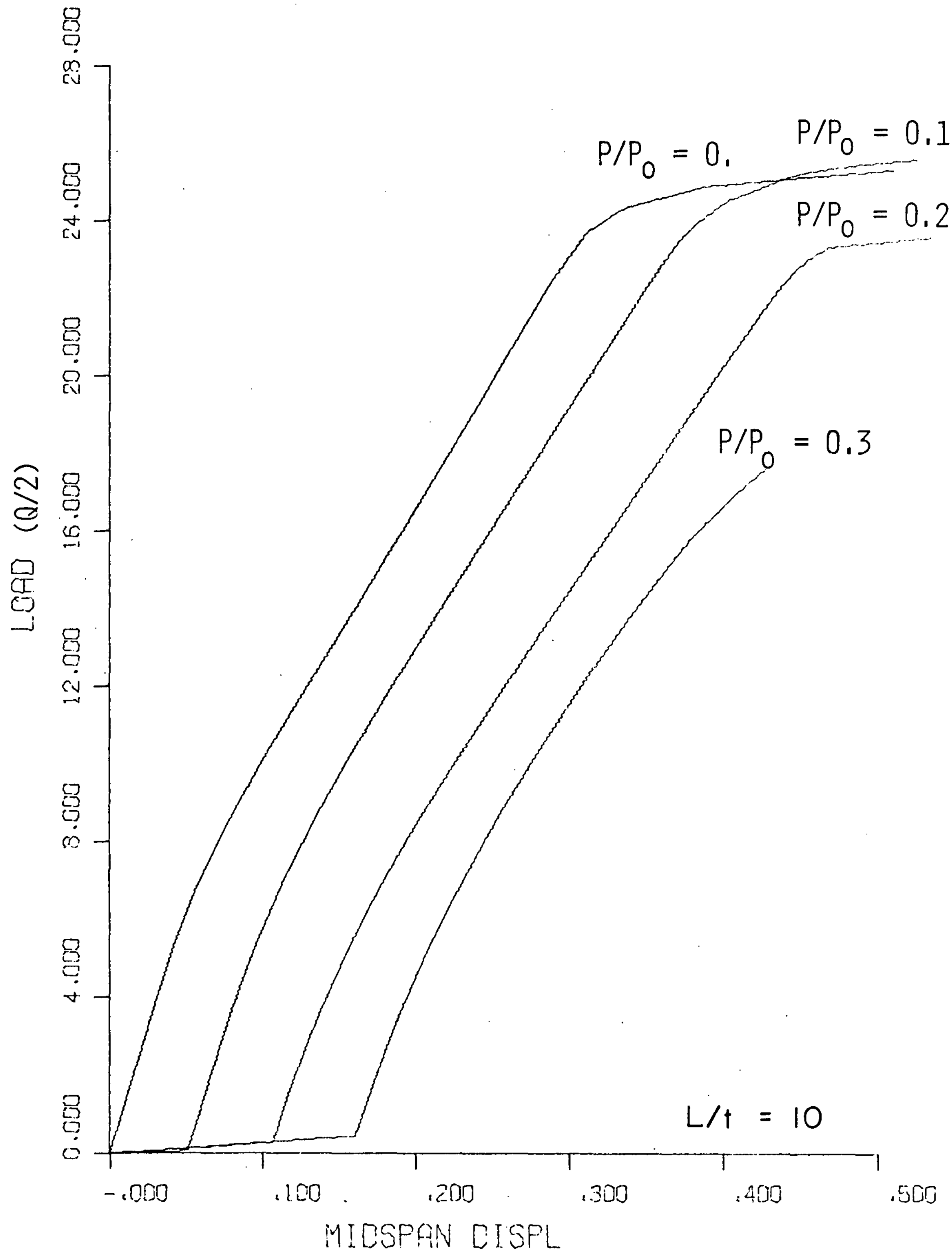

Fig. 18 Load Deflection Curves for Reinforced Concrete Beam Column

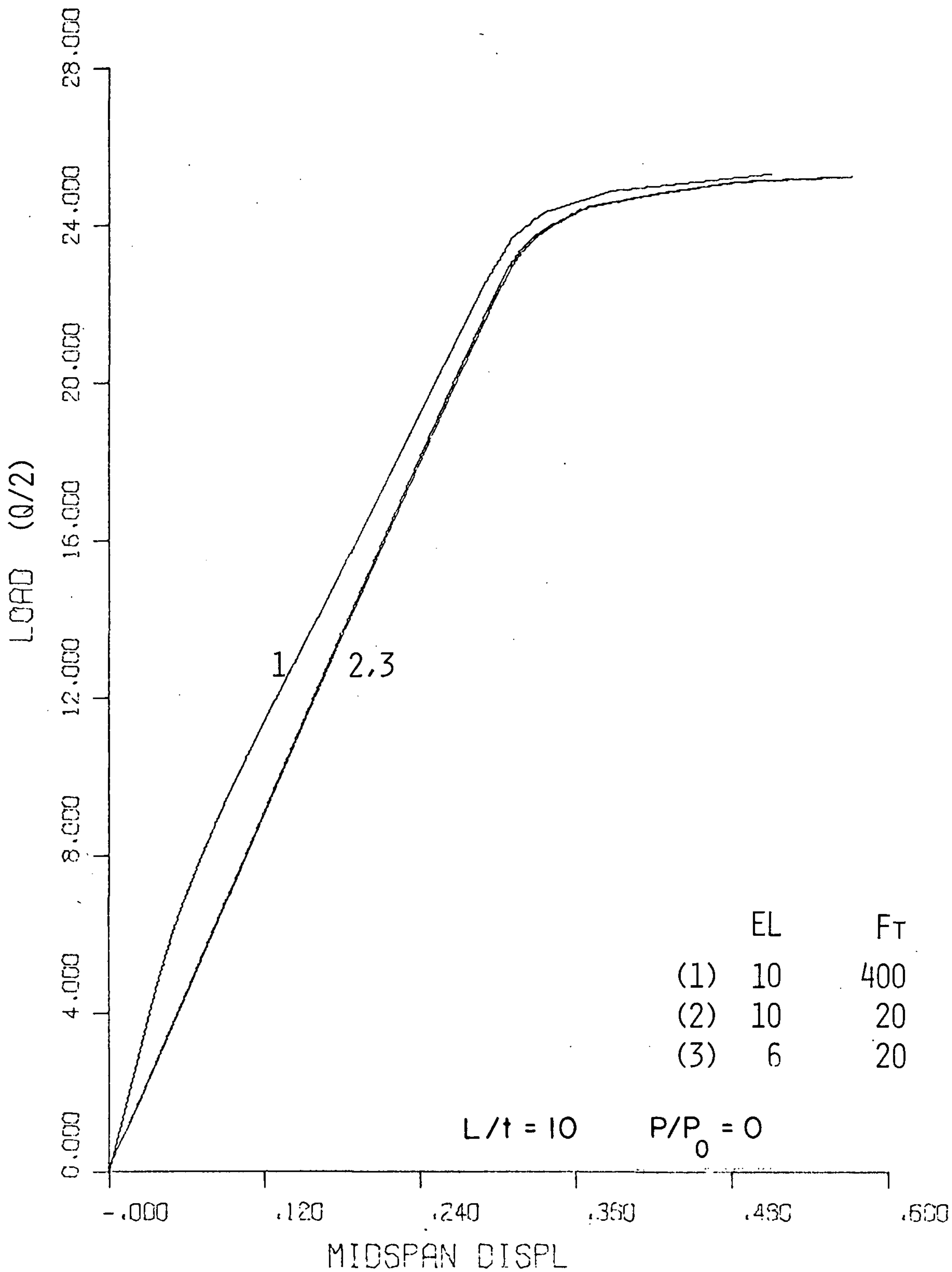

Fig. 19 Load Deflection Curves for Reinforced Concrete Beam Column

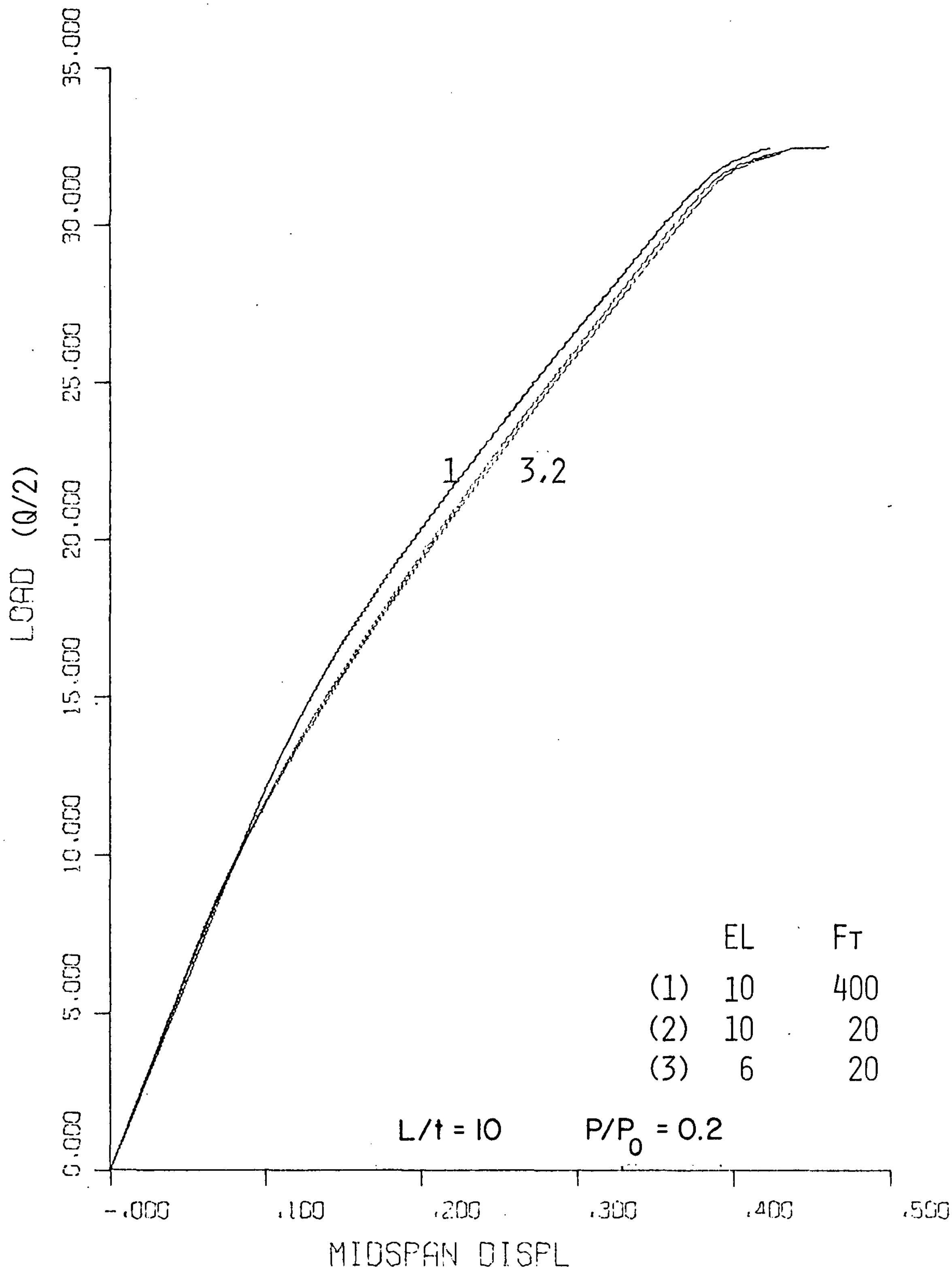

Fig. 20 Load Deflection Curves for Reinforced Concrete Beam Column

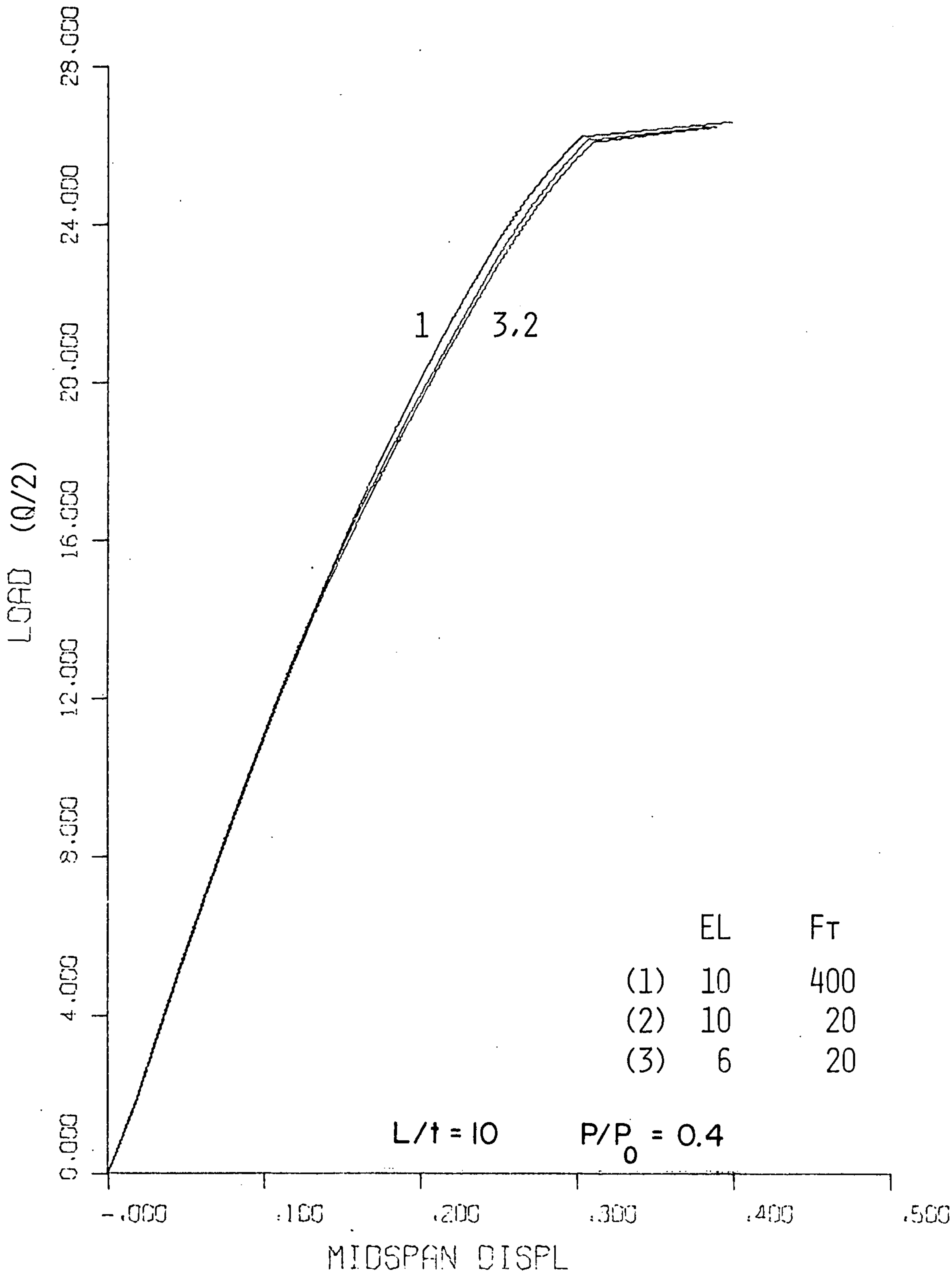

Fig. 21 Load Deflection Curves for Reinforced Concrete Beam Column

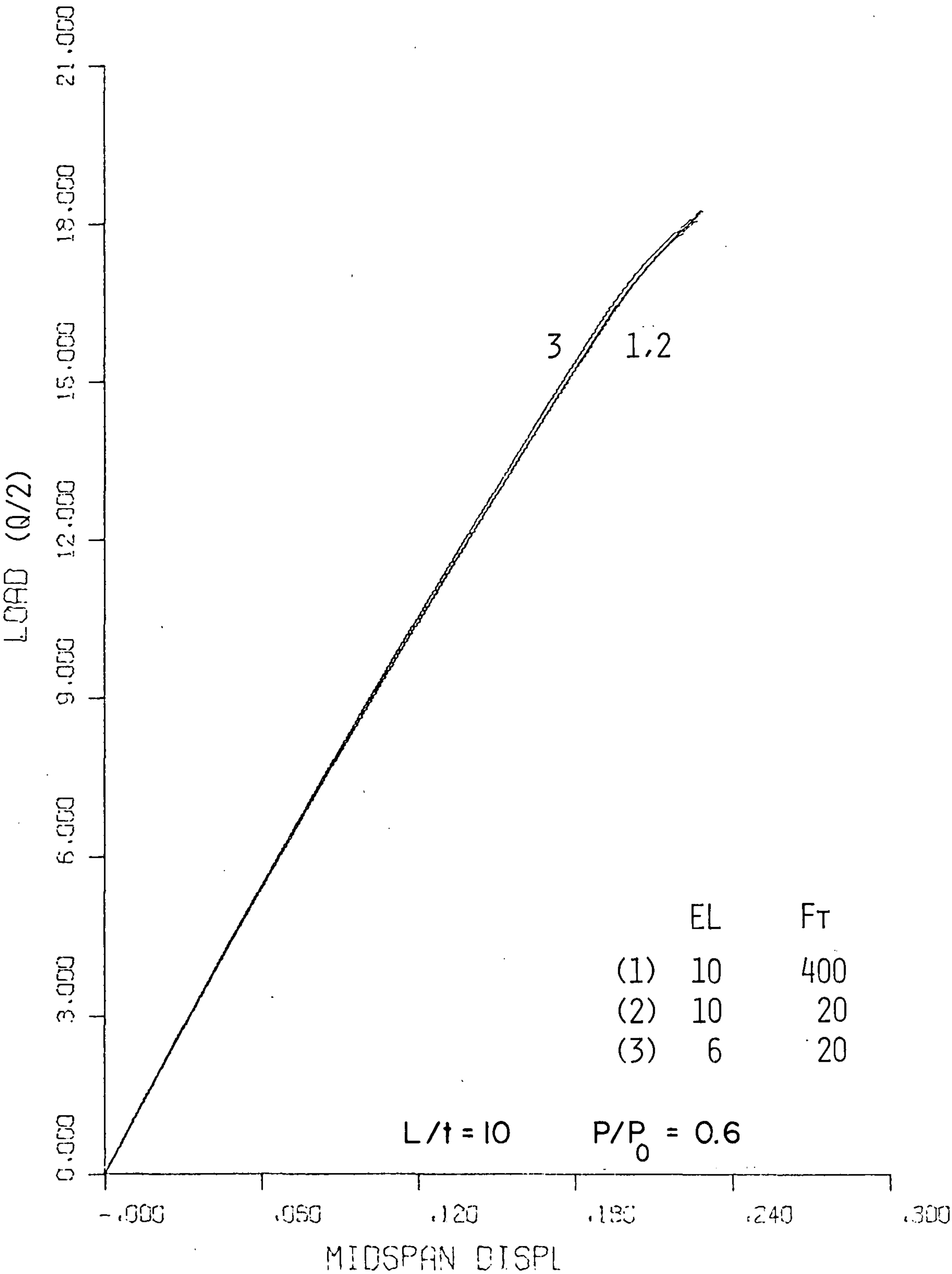

Fig. 22 Load Deflection Curves for Reinforced Concrete Beam Column

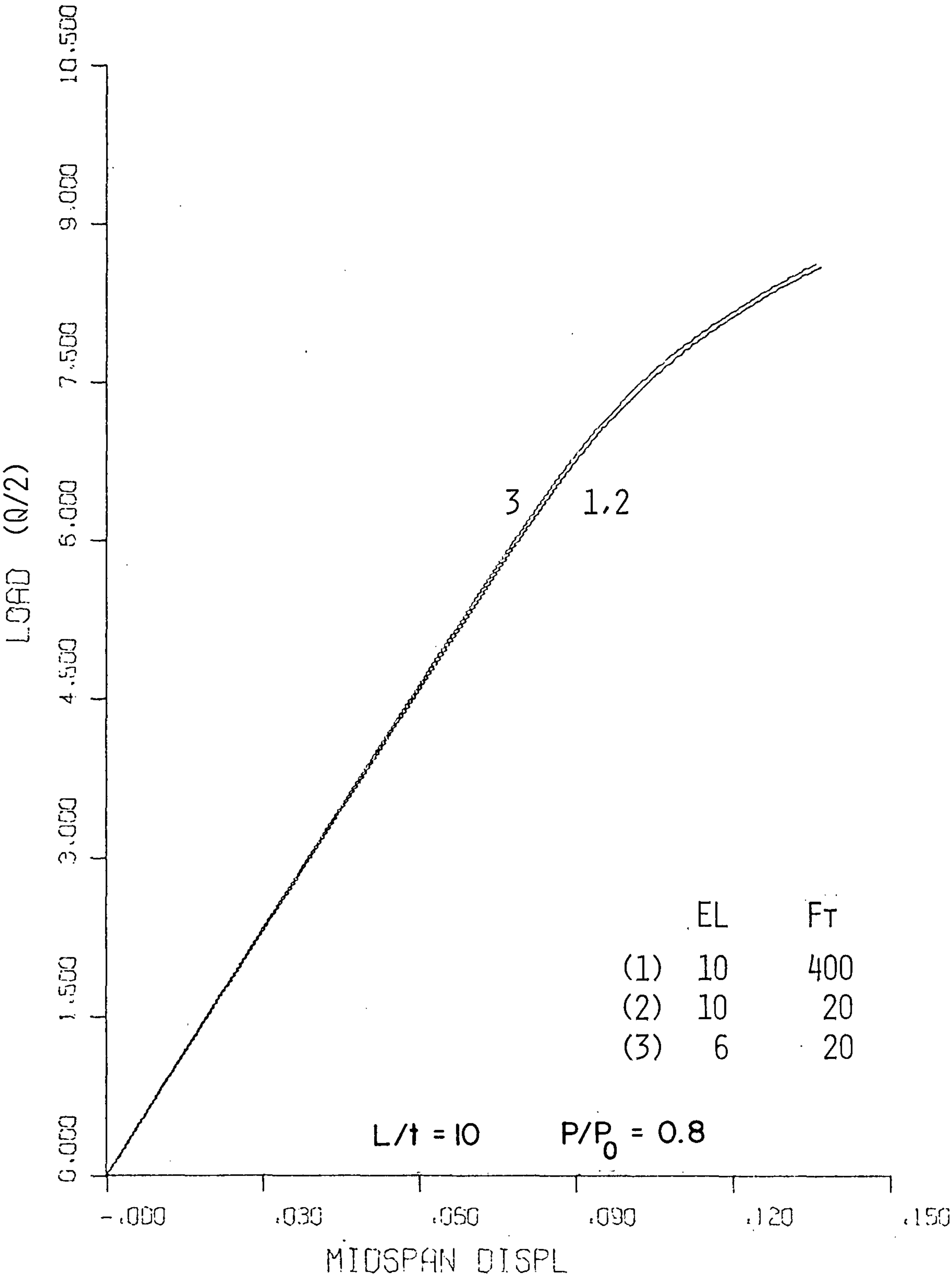

Fig. 23 Load Deflection Curves for Reinforced Concrete Beam Column

7. NOMENCLATURE

P	= An axial load
Q	= Lateral load
δ_{ai}	= Displacements at mid-segment of sebment i - CDC
M_{qi}	= Moment at mid-segment of segment i - CDC
θ_o	= Initial slope - CDC
δ_i	= Displacement at the end of segment i - CDC
ϕ_{ai}	= Average curvature in segment i - CDC
ρ	= Length of a segment - CDC
M_i	= Moment at end of segment i - CDC
θ_i	= Slope at end of segment i - CDC
q	= A uniform load - CCC
M	= Bending moment - CCC
y	= Lateral deflection - CCC
x	= Length along beam column - CCC
ϕ	= Curvature - CCC
u	= Axial displacement - FEM
$\{\delta^e\}$	= Elemental displacement vector - FEM
$\{\alpha\}$	= Constants - FEM

$[Q],[C]$ = Connection matrices - FEM

$\{\sigma\}$ = Stress vector - FEM

$\{\epsilon\}$ = Strain vector - FEM

N = Axial force - FEM

$[K]$ = Stiffness matrix - FEM

$\bar{A}$ = A generalized area - FEM

$\bar{S}$ = A generalized statical moment - FEM

$\bar{I}$ = A generalized moment of inertia - FEM

J = Number of layers in an element - FEM

ℓ = Element length - FEM

L = Beam column length

$[K_G]$ = Geometric stiffness matrix - FEM

E = Initial modulus of elasticity - FEM

σ_1 = A yield stress - FEM

$\bar{M}$ = A Ramberg-Osgood parameter

$\bar{N}$ = A Ramberg-Osgood parameter

f'_c = Concrete cylinder strength

t = Depth of concrete beam column

F_t = Allowable tensile stress

r = Radius of gyration about bending axis

P_y = Axial yield load of a column

f''_c = $0.85\ f'_c$

P_o = $f''_c\ A_c + f_y\ A_s$

A_c = Concrete area

f_y = Yield stress

A_s = Steel area

M_o = Balanced moment for a concrete beam-column

Q_o = Ultimate load of a concrete beam with no axial load

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9. ACKNOWLEDGMENTS

This study was conducted in the Department of Civil Engineering at Fritz Engineering Laboratory, Lehigh University, Bethlehem, Pennsylvania.

The authors wish to extend their thanks to Mr. Negussie Tebedge for his explanation of those portions of his doctoral work used here and to Dr. W. F. Chen and Mr. A. C. T. Chen for their cooperation with details of their previous work used as comparative examples.

Thanks is also extended to the staff of the Lehigh University Computing Center for their cooperation and to Mrs. Ruth Grimes who typed the manuscript.